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Stochastic Approach for Obtaining Seismic Wave Paths in Random Heterogeneous Media

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Synopsis

A stochastic approach is proposed for obtaining two-point seismic wave paths in a three-dimensional random heterogeneous media, by deriving and solving analytical equations and showing some numerical examples. For cases where the media include random heterogeneities characterized by statistic functions, we cannot deterministically obtain seismic wave paths, but stochastically we define a stochastic tube around a reference ray. The stochastic tube connects a source point and an observation station, consists of wave paths of the same type as the reference ray, and represents the probabilistic multiple paths. We derive and analytically solve a stochastic differential equation to represent the stochastic tube. The stochastic tube, thus obtained, indicates a probability density function of the seismic wave paths. The stochastic tubes for first arrival P-wave paths in three-dimensional random heterogeneous media are numerically calculated to illustrate a few examples using stochastic calculus.

Keywords: stochastic tube, random heterogeneity, ray tracing, traveltime tomography, Brownian motion

1. Introduction

Theoretical approaches for calculating seismic waves using the high-frequency approximations provide indispensable geophysical tools for investigations of the interior of the Earth, especially for information about arrival times and ray paths. Ray theory is applicable when the wavenumber k of the high-frequency seismic wave is large compared with the scale a of the heterogeneity in the structure; that is $ak > 10$ or the condition that a is large compared to the Fresnel zone (Aki and Richards, 1980). Ray tracing methods aim at deterministically finding robust and accurate ray paths and Julian and Gubbins (1977) investigated their effectiveness by comparing two of the major seismic ray tracing methods. One is a shooting method, where the initial direction of a ray shot from a source is changed until the ray arrives at an observation station. The other is a bending method, in which the ray is found by using Fermat's principle so that it connects the source and the station with the shortest traveltime. Um and Thurber (1987) proposed an approximate approach to find the path with the shortest traveltime by application of the bending method. But using either method it is difficult to find the true ray paths in a very complicated velocity structure, because the shooting method requires a tremendous number of trials and the bending method often falls into a local minimum of the traveltimes. To overcome these problems, for example, Nishi (2001) recently proposed a hybrid calculation scheme for highly heterogeneous three-dimensional velocity structures.

Traveltime seismic tomography using the first arrival phase is a widely used method to study the Earth's interior. The actual velocity structure usually is not completely understood and sometimes characterized by mean velocity and statistic functions to represent the random heterogeneities. For such the structures, a ray theoretical relation between the autocorrelation function of traveltimes and that of random media can be presented (Műller et al., 1992), however, the exact two-point ray is available, for example, by using ray perturbation theory (Farra and Madariaga, 1987; Snieder and Sambridge, 1992) only if we know the exact velocity structure. It is difficult to represent comprehensive seismic wave paths, which may be the first arrivals, if we do not know the exact heterogeneities but know only statistic functions characterizing the heterogeneities. In this study, our motivation is to stochastically determine the seismic wave paths instead of the exact ray paths in the random heterogeneous media. We apply the high-frequency approximation and a stochastic approach. A stochastic tube is defined as a bundle of stochastic paths, which are diffused by random heterogeneities in the neighborhood of a reference ray path. A stochastic differential equation for the stochastic tube is derived and analytically solved. Some numerical examples are shown using our stochastic method.

2. Stochastic tube tracing method in a random heterogeneous medium

We construct a random heterogeneous medium by using a background structure and then applying random perturbations (Table 1). Seismic rays are available in the background structure for $ak > 10$ and $kL > 10$ (Aki and Richards, 1980), where a, k and L are the correlation distance of the anomalies, the wavenumber and the travel distance, respectively. A first arrival seismic wave path connecting a source point and an observation station is given by a ray path. When the background structure is perturbed, the ray path is also perturbed. The exact ray would be distinctly determined, however the traveltime may not provide the first arrival (Roth et al., 1993). Then we consider the probable wave paths, which are random paths perturbed around the original ray path, a reference ray path. The probable paths are constructed by considering the diffusion (or perturbation) about the reference ray path, which is associated with perturbations to the background structure (Table 1). The paths are not always rays. We express the probable paths involving the diffusion, as stochastic paths in the vicinity of the reference ray path, by use of a stochastic approach, while the reference ray path is conventionally determined as a ray path in the background structure.

The stochastic paths are spatially diffused (or perturbed) around the reference ray path. We assume that the wave type of the stochastic path is identical with that of the reference ray. The traveltime of the stochastic path, which is calculated along the path, is the same as that of the reference ray or shorter. For example, the stochastic paths for the reference ray of a first arrival P-wave are also the paths of the P-wave, and they arrive at an observation station at the same times as the P-wave ray or earlier. If the structure does not include random heterogeneities, the stochastic paths are identical with the reference ray path because there is no diffusion. We refer to the bundle of the stochastic paths for the reference ray path as a stochastic tube (Table 1), which is obtained by deriving and solving a stochastic differential equation in the following section.

2.1 The stochastic tube model

We suppose d stochastic paths in the neighborhood of a reference ray path, allowing their loci to overlap, where d is the dimension of the stochastic process and the total number of samples in the present problem. We refer to a bundle of d stochastic paths as a d-dimensional stochastic tube for the reference ray. The method of obtaining the stochastic tube is called stochastic tube tracing. In the neighborhood of the reference ray path, the stochastic tube is represented by use of a stochastic differential equation.

To derive the stochastic differential equation, let us consider the cross-section of the stochastic tube perpendicular to the reference ray path (Fig.1a). A position vector along the reference ray path at time $t \in [0, T]$ is given by

$$x = \int_0^t v(t')n dt' \qquad (1)$$

where n is the tangent vector along the reference ray path, $v(t')$ substituted for $v(x(t'))$ is the seismic velocity at $x(t')$ and T is the traveltime at the receiver station (Fig.1b).

Table 1 Construction of random heterogeneous medium and stochastic path (tube)

[random heterogeneous medium]	=	[background]	+	[random perturbations]
[stochastic path (tube))	=	[reference ray path]	+	[diffusion]
X^i_{stc}	=	x	+	X^i

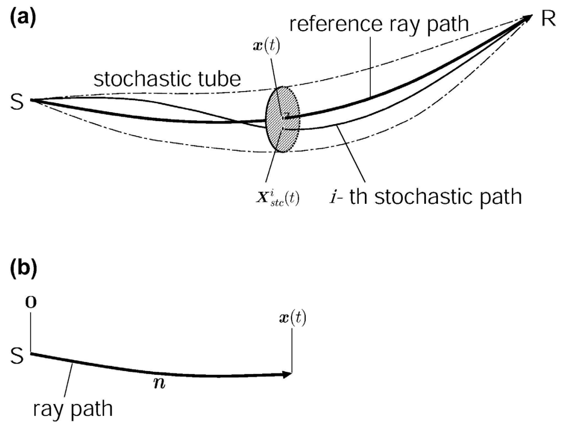

Fig. 1 (a) Geometry of a stochastic tube. A seismic source and a receiver station are represented as S and R, respectively. The reference ray path connects S and R. $X^i_{stc}(t)$ is the intersection of the i-th stochastic path in a cross-section perpendicular to $x(t)$ in the reference ray path for time t. A stochastic tube is constructed by the bundle of d stochastic paths. (b) Position vector $x(t)$ in the ray path. S is both the source point and the origin of the vector. n is the tangent vector along the ray path.

We also suppose the source is at the origin of the vector, $x(0) = 0$. The cross-section of the stochastic tube, with respect to $x(t)$ in N-dimensional real space, indicates spatial differences between the reference ray path and the stochastic paths, and is supposed to be expressed as a d-dimensional stochastic process

$$X(t) = \{X^i(t);\ 1 \le i \le d\} \ \in\ R^{d\times N}, \tag{2}$$

where N is either 2 or 3, the i-th component $X^i(t)$ is an N-dimensional vector measured from $x(t)$ and

$$x(0) = x(T) = 0 \tag{3}$$

because the source point of the reference ray path and stochastic paths is common and the receiver also is common (Fig.1a). $X^i(t) = 0$ means that the i-th stochastic path intersects with the reference ray path at time t of the reference ray. Then the stochastic tube $X_{stc}(t)$ for time t of the reference ray is

$$X^i_{stc}(t) = x(t) + X^i(t) \tag{4}$$

for $1 \le i \le d$ (Table 1), where

$$X_{stc}(t) = \{X^i_{stc}(t)\ ;\ 1 \le i \le d\} \ \in\ R^{d\times N}, \tag{5}$$

$$X^i_{stc}(0) = x(0) = 0 \tag{6}$$

and

$$X^i_{stc}(T) = x(T). \tag{7}$$

2.2 Derivation and solution of the stochastic differential equation

In an arbitrary cross-section perpendicular to the reference ray path, we formulate the stochastic differential equation for $X_{stc}(t)$ assuming that the stochastic tube is closed on both ends, as is indicated in equations (6) and (7) and illustrated in Fig.1(a). Equation (7) also means that the stochastic distribution of $X_{stc}(t)$ is given by $x(T)$ when time t becomes T, then this stochastic process is called positive recurrent in probability theory. The stochastic paths in the heterogeneous medium are assumed to be represented by a diffusion process. Perturbation of the stochastic path would result in the construction of a tubular shape around the reference ray path. Therefore, the infinitesimal change of $X^i_{stc}(t)$, which is $dX^i_{stc}(t)$, is supposed to be equivalent to the summation of three factors: (i) the infinitesimal change $dx(t)$ of the reference ray path, (ii) the convergence of the stochastic tube so that every stochastic path finally arrives at the receiver station and (iii) diffusion during the time interval $t \rightarrow t + dt$.

We can analytically represent these three factors. From equation (1), (i) $dx(t)$ is rewritten into

$$v(t)\ dt, \tag{8}$$

where $v(t)$ is the seismic wave velocity vector along n at $x(t)$. The term (ii) of the convergence of $X^i_{stc}(t)$ is written as

$$\frac{1}{T-t}\left\{\int_0^t v(t')dt' - X^i_{stc}(t)\right\}dt, \tag{9}$$

where we assume any stochastic path linearly converges to the reference ray path as time goes to T. This assumption is the simplest model for convergence and geometrically reasonable. To give the diffusion term (iii), we use Brownian motion because it can properly represent the perturbed paths in a medium including velocity perturbations. During this infinitesimal change, a diffusion coefficient α in $X_{stc}(t)$ and d'-dimensional Brownian motion $B(t)$ express this diffusion term as

$$\alpha\left(X_{stc}(t)\right)dB(t) \quad \left(=\sum_{k=1}^{d'}\alpha^i\left(X^i_{stc}(t)\right)dB^k(t)\right), \qquad (10)$$

where d' is the dimension of the stochastic process, the Brownian motion, and the coefficient $\alpha^i(X^i_{stc}(t))$ (= α^i) is the i-th element in

$$\alpha^i = \{\alpha^i; 1 \le i \le d\} \in R^{d\times d'}. \qquad (11)$$

We also define the d'-dimensional Brownian motion

$$B(t) = \{B^k(t); 1 \le k \le d'\} \in R^{d'\times N}. \qquad (12)$$

The Brownian motion is assumed to move in any direction independent of the next time instant. In the stochastic process $B^k(t)$, differences $\{B^k(t_j)-B^k(t_{j-1})\}_{1\le j\le n}$, where $0 = t_0 < t_1 < \cdots < t_n = T$, are independent of each other and depend on a Gaussian distribution.

Hence, putting equations (8)–(10) together, we obtain a stochastic differential equation for $X^i_{stc}(t)$;

$$dX^i_{stc}(t) = \sum_{k=1}^{d'}\alpha^i dB^k(t) + b^i dt \qquad (13)$$

for $1 \le i \le d$ and $0 \le t < T$, or

$$dX_{stc}(t) = \alpha\, dB(t) + b\, dt \qquad (14)$$

for $0 \le t < T$, where

$$b^i = \frac{1}{T-t}\left\{v(t)T + \int_0^t \left(v(t') - v(t')\big|_{t'=t}\right)dt' - X^i_{stc}(t)\right\} \qquad (15)$$

is called the drift coefficient. Since α and b satisfy the Lipschitz condition for $1 \le t < T$, equation (13) has a unique solution for $X^i_{stc}(t)$ by use of Ito's formula (Ikeda and Watanabe, 1989; Funaki, 1997), which corresponds to a composed function of the differential formula. The uniqueness and the Lipschitz condition are discussed in Appendix. Since the equation is similar to that of a Brownian bridge (e.g. Karatzas and Shreve, 1991), the structure of the solution is also expected to resemble that of the Brownian bridge. Consequently, we have the unique solution to equation (13),

$$X^i_{stc}(t) = X^i_{stc}(0) + \int_0^t v(t')dt' + (T-t)\sum_{k=1}^{d'}\int_0^t \frac{\alpha^i}{T-t'}dB^k(t') \qquad (16)$$

for $0 \le t < T$. Substituting equations (1), (4) and (6) for equation (16), we have

$$X^i(t) = (T-t)\sum_{k=1}^{d'}\int_0^t \frac{\alpha^i}{T-t'}dB^k(t') \qquad (17)$$

for $0 \le t < T$.

It is noteworthy that the stochastic integral with the Brownian motion differs from a Stieltjes integral and is evaluated by Ito's stochastic integral (Ikeda and Watanabe, 1989). The stochastic process $X_{stc}(t)$ satisfies the condition that a physical state immediately following time t depends only on the current state on t and is not affected by any states immediately before t. This is known as a Markov process. Therefore, this irreversible process generates the stochastic paths only from the source to the receiver, and does not guarantee the same paths from the receiver to the source, as in conventional deterministic ray paths. However, if the exact ray satisfies the condition for the stochastic path, the stochastic tube includes the element which is reciprocal. When $t \uparrow T$, we have an expected value (expectation value)

$$E\left[\left|X^i_{stc}(t) - x(t)\right|^2\right] = \left|(T-t)\sum_{k=1}^{d'}\int_0^t \frac{\alpha^i}{T-t'}dt'\right|^2 \to 0 \qquad (18)$$

thus the stochastic tube converges to the receiver station ($X^i_{stc}(t) \to x(t)$) and equation (16) for $t \uparrow T$ is consistent with equation (7). $P(X^i(t))$, the probability of a stochastic path included in $X^i(t)$, is estimated by the ratio of the number of samples included in $X^i(t)$, to the total number of samples d of the stochastic process $X(t)$ in the arbitrary cross-section. When the medium is homogeneous with no diffusion (X = 0), equation (4) implies that the stochastic tube is identical with the reference ray path itself ($X_{stc} = x$), therefore indicating that our stochastic approach would become deterministic.

There is no mathematical connection between the stochastic path and ray. Equations (9) and (10) that are applicable to any possible wave path, cannot be expressed in Hamiltonian perturbation in the ray perturbation theory (Farra and Madariaga, 1987).

3. Numerical examples of stochastic tubes

By use of stochastic tube tracing, we show some numerical examples that obtain seismic wave paths connecting a source point and a receiver station through a three-dimensional heterogeneous medium ($N = 3$). The medium consists of a background structure and its perturbation, as indicated in Table 1. The background structure of the medium has a seismic velocity linearly increasing with depth. The perturbation on this background structure includes anomalies with a correlation distance a and a velocity perturbation of ε, which indicate their size and corresponding amplitude, respectively. We suppose ε is less than 0.1, because the actual perturbation of seismic wave velocities has been at most about 10% irrespective of any scale length in the Earth (Wu and Aki, 1988). The autocorrelation function (ACF) characterizes the isotropic perturbation with a and ε. We adopt an exponential ACF, $R(x)$, with a power spectral density function given by

$$|F(R(x))| = \frac{8\pi\varepsilon^2 a^3}{(1+a^2m^2)^2}, \quad (19)$$

where F denotes a Fourier transformation and m is the wavenumber for the model space x (Frankel and Clayton, 1986; Sato and Fehler, 1998). $R(x)$ is finally given by an inverse Fourier transformation for $F(R(x))$, which is equation (19) multiplied by an initial white noise $exp(i\theta(m))$, where $\theta(m)$ takes random values in the range of 0 to 2π.

The random heterogeneity determines how the stochastic tube diffuses, and the diffusion coefficient should be analytically governed by the random heterogeneity. However, the condition for the stochastic tube that the traveltime should be equal to or no longer than that of the reference ray makes the analytical determination difficult. The coefficient is numerically obtained through our calculations. In this three-dimensional random heterogeneous medium, we attempt d-dimensional stochastic tube tracing for a first arrival P-wave. The total number of samples in the stochastic process is d = 1,000, which is statistically sufficient.

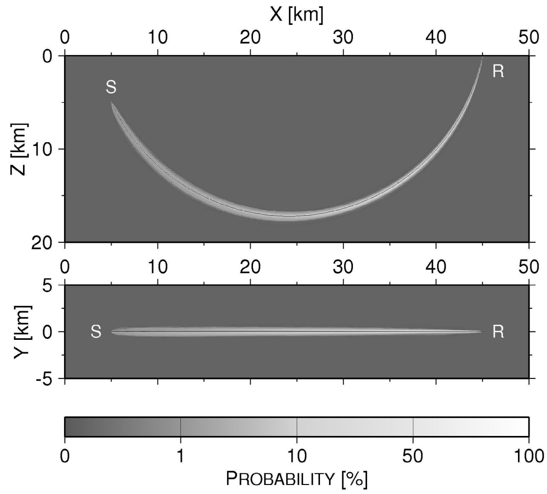

Fig. 2 Stochastic tube for a reference ray of the first arrival P-wave projected onto the XZ-plane and the XY-plane. X and Y are horizontal coordinates and Z is the vertical, which form a left-hand system. The probability indicates the ratio of stochastic paths through each 0.05 × 0.05 km^2 square in the projected planes, to the total number of samples (d = 1,000). The light region indicates high probability and the gray low probability. S and R are the source point (5, 0, 5) and receiver station (45, 0, 0), respectively. The background velocity is $0.25Z + 1$ km/s and its perturbation is characterized by the exponential ACF with a = 0.5 km and ε = 0.01. The solid line in the stochastic tube is the reference ray path.

3.1 Simple examples

Figure 2 shows the stochastic tube projected onto the XZ-plane and the XY-plane, where the medium has random heterogeneity with a = 0.5 km and ε = 0.01 and the background velocity is $0.25Z + 1$ km/s. The stochastic tube becomes gradually wider with the extension of the path length measured from the source, while the tube begins to converge to the reference ray as it approaches the receiver station. The width for downgoing tube near the source seems to be larger than that for upgoing tube near the receiver station. This is consistent with the property of geometrical spreading. The probability in any region indicates the ratio of the number of stochastic paths in the region to the total number of samples d. High probabilities are expected in the neighborhood of the reference ray path.

Figure 3 shows cross sections of each stochastic tube at the turning point of the reference ray path for different *a* (= 0.5, 1.0 and 2.0 km) and *ε* (= 0.005, 0.01 and 0.02), and the corresponding diffusion coefficients are indicated in Fig.4. The tubular width becomes large with decreasing *a* and increasing *ε*. Narrow tubes for small *ε* affirm the guess that, if the medium has no perturbations and the seismic wave is not affected by the heterogeneities, the diffusion term along the reference ray path would be zero and our stochastic method would be identical with the deterministic one, as well as the analytical form in equation (4) for $X^{i}(t) \to 0$. The cross-sections are elliptical shapes having major axes in the vertical direction, because the background structure contains a vertical velocity increase and its three-dimensional isotropic perturbation results in

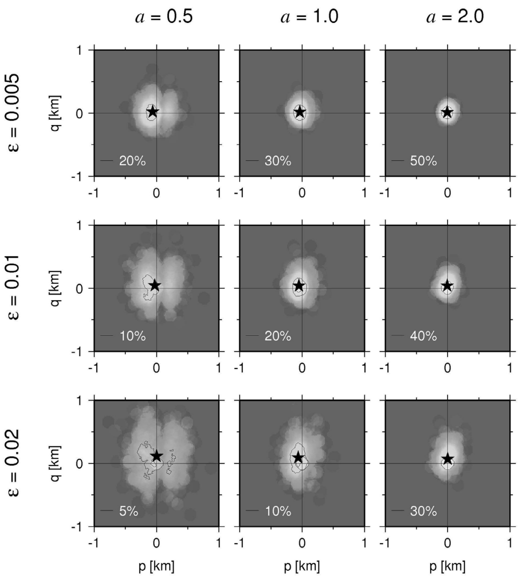

Fig. 3 Cross-sections of the stochastic tubes for each structure. The background structure, the source point and the receiver station are the same as in Fig.2. Anomalies are calculated for perturbations of *a* = 0.5, 1.0 and 2.0 km and *ε* = 0.005, 0.01 and 0.02. The cross-section planes are perpendicular to the reference ray path at the turning point. *p*- and *q*-axes include transverse and vertical components from the receiver point of view, respectively. In these cases, unit vectors *p*/|*p*| and *q*/|*q*| are (0, 1, 0) and (0, 0, -1), respectively. The origin is identical with the reference ray path. The probability for an arbitrary point is estimated by the ratio of the number of stochastic paths included in the arbitrary point centered circle of 0.1 km radius, to the total number of samples (*d* = 1,000). Stars show the expectation values in each cross-section. The grayscale of probability is identical with that in Fig.2.

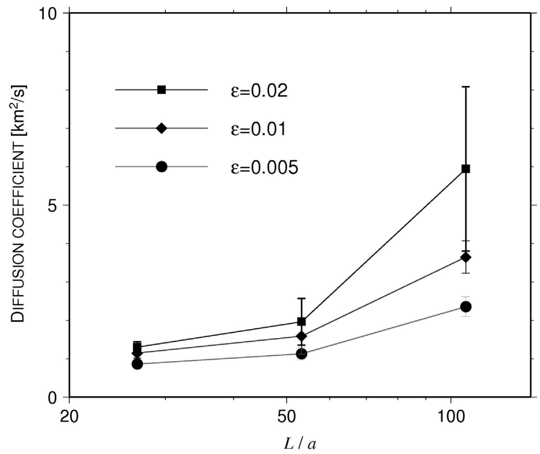

Fig. 4 Diffusion coefficients of stochastic tubes for each media of *a* and *ε* indicated in Fig.3. *L* is the travel distance of the reference ray. The diffusion along the vertical component (or *q*-axis) is always larger than lateral component (or *p*-axis). The bars show differences of the diffusion between the major axis and the minor axis of the elliptical shape in the cross-section.

stronger diffusion across the vertical component than across the horizontal component. The highest probability area in Fig.3 is similar to the reference path that is at the origin of the illustrated coordinates. In the case of a highly diffused tube of *a* = 0.5 and *ε* = 0.02, there exist some local regions with high probabilities. All expectation values, shown by stars in the cross-sections, seem to have a slight bias toward the +*q*- and −*p*-directions from the origin. We use the exponential ACF (19) to construct the heterogeneities by changing *a* and *ε*, while the initial complex white noise $exp(i\theta(m))$ for *m* is identical for the different cases. This identical white noise controls a similar tendency towards either slow or fast velocities of the spatial heterogeneities, then the diffusion patterns tend to be similar.

The random heterogeneous medium is constructed with statistical parameters *a* and *ε* and arbitrary white noise. We are then able to give infinitely probable structures characterized by the ACF. Even though the different deterministic structures are chosen by a common ACF but different initial white noise, the properties of the stochastic tubes show little differences in the present cases.

3.2 Triplication of a reference ray

We consider another case where the background structure is not as simple and has sudden but smooth increases of velocity beneath a certain depth (Fig.5). At certain epicentral distances three distinct traveltimes exist, so there are three ray paths, a triplication. We choose an observation station corresponding to the crossing point of the branches of traveltime curve, so the first two arrivals have the same traveltime followed by the third arrival. The three multiple ray paths are indicated in the left portion of Fig.5 with solid lines. By perturbing this background with an exponential ACF of a = 1.0 km and ε = 0.01, we construct a new random heterogeneous medium. The total number of samples in the stochastic process is d = 3,000. The stochastic tubes for the first arrival of the P-waves in this structure are shown in Fig.5. Two major stochastic tubes with widths of ~2 km are seen at depths of about 10.5 km and 12.5 km around the turning points. These calculated major stochastic tubes mean the existence of two probable roots. These tubes seem not to be completely separated, but connected by a low probability region, for example at depths of 11.5–12.0 km of the cross section around the turning point. The upper and the lower major stochastic tubes have higher probabilities and are wider than the middle bridge connecting them. These two stochastic tubes are constructed around the ray paths having the earlier traveltime, while the middle bridge is around the late arrival ray path.

4. Discussion

We will firstly discuss the variable widths of the stochastic tubes depending on the properties of heterogeneities, secondly the dimension of stochastic process d for calculation and expectation values, and thirdly the application of the stochastic tube tracing to traveltime tomography.

As shown in Figs 3 and 4, the stochastic tube becomes wider by decreasing the anomaly size a and increasing its amplitude ε for the same source-observatory set. The stochastic paths are the same type as the reference ray, and in the present cases they are the paths of the first arrival P-wave. When ε becomes large, there is a strong tendency for traveltimes of diffused paths to be either much faster or slower, than the case of small ε. This means that it is possible that paths with faster traveltimes become first arrival paths, even with a negligible increase of path length. When a becomes small, the medium includes many small anomalies and there are many choices for the diffused paths to pass through positively perturbed anomalies (fast regions) and avoid passing negatively perturbed regions. The travel distance of the largely diffused path can be reduced by efficient path selection keeping the large diffusion, then the traveltime becomes short for the stochastic path. In the both cases of small a and large ε, the stochastic paths are able to choose highly perturbed (diffused) paths and the stochastic tube has a wide tubular shape. Witte et al. (1996) stated that the diffracted waves have earlier traveltimes, compared to the exact rays, as a decreases and ε increases. Then, the wide stochastic tube may not only show the possibility of large perturbations from the reference ray but also represent the paths of diffracted waves. On the other hand, small ε means smoother media, producing very narrow stochastic tubes. The infinitesimally narrow stochastic tube in such media is identical with the reference ray, as mentioned before, which is then consistent with the ray

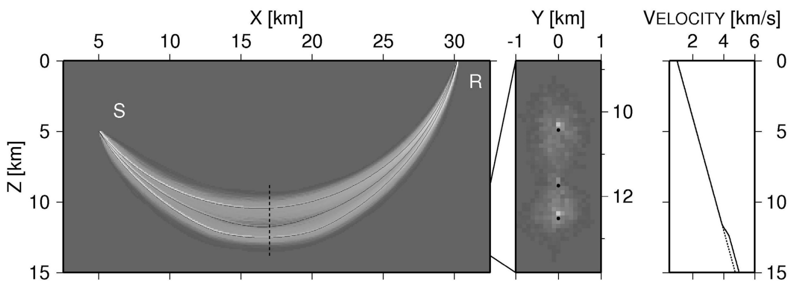

Fig. 5 Stochastic tube for the reference ray in the case of a triplication. The left figure is the tube projected onto the *XZ*-plane, where solid curves are multiple ray paths of the triplication. The middle figure is the magnified tube crossing the *YZ*-plane, which is also illustrated with a vertical broken line in the left figure. Small filled circles indicate the cross sections of the multiple reference ray paths. In the right portion, the background velocity structure is shown with a solid line, indicating that the velocity suddenly but smoothly increases beneath a certain depth. The perturbation is characterized by the exponential ACF with a = 1.0 km and ε = 0.01. The total number of samples is d = 3,000. The grayscale of probability is identical with that in Fig.2.

method designed for smooth media. In the case of very large a, for example $L/a << 10$, the anomalies should be considered in the background.

Numerical examples clearly show the actual stochastic tubes in heterogeneous structures characterized by an AFC with static parameters, a and ε. To obtain the probability of a stochastic tube in detail, many samples in the stochastic process are required, which may be numerically intensive. We choose d = 1,000 for simple cases in the present study (Figs 2 and 3), but we confirm that cases of more than d = 100 are statistically adequate to illustrate the probability for simple stochastic tracing and no significant difference is found among these cases. Even when multiple paths exist in the background structure, the present method effectively represents the probability of the stochastic tube in the vicinity of each multiple path (Fig.5). The second case with the more complicated structure, however, preferably requires a higher dimension of the stochastic process, such as $d >> 100$.

In the case of a triplication, the late arrival ray path is clearly separate from the paths of the first arrival, when we consider only the background structure. However, as indicated in Fig.5, when the background structure is perturbed, we can find a low probability existence of the first arrival stochastic paths even around the late arrival ray path, in addition to the expected high probabilities around the first arrival ray paths. The results indicate that in a randomly perturbed structure there might exist first arrival multiple paths that cannot be predicted in the background structure and are shown by the stochastic method. Such multiple paths could be often expected beneath heterogeneous structure, hence we probably should consider stochastic paths in detecting large background structures, such as velocity boundaries, when the properties of the corresponding anomalies, a and ε, are small and large, respectively.

Calculation of expectation values of the stochastic tube provides useful information, and the probable corresponding seismic wave path would be uniquely decided even in random heterogeneous media. The expected regions almost correspond to the reference ray paths because the ensemble of heterogeneities in random media is statistically zero, indicating that ray paths obtained in the background structure are generally valid for the expected path detection. When the tube occupies a large region or the reference ray involves a triplication in the background structure, however, the expectation cannot always identify a highly expected path (e.g. Fig.5). Hence, the expectation values should be used to pinpoint narrow tubes, in which some high probability regions do not locally exist.

The width of the tube indicates the area of influence, through which the seismic wave paths, of the same type as the reference ray, should be affected in traveltime calculation. The area is expressed by the probabilities of the stochastic tube and then a stochastic density function of the stochastic tube indicates how the area affects the traveltimes of the seismic paths. This function provides a convenient way to apply stochastic tube tracing to linear traveltime tomography where the random heterogeneities are already characterized. The observation equation is given by

$$\Delta T_j^i = \int_{path} \Delta s \, dx_j^i \ , \tag{20}$$

where ΔT^i_j is a residual between an observed traveltime and a traveltime of the j-th source-observatory set for the i-th stochastic path x^i_j ($1 \le i \le d$) and Δs is a slowness perturbation. We discretize equation (20) to get

$$\Delta T^i_j = G^i_j \Delta s, \tag{21}$$

where G indicates the discretization of the corresponding Fréchet kernel operator. If we consider only the j-th reference ray, the observation equation is

$$\Delta T_j = G_j \, \Delta s. \tag{22}$$

If variation of $|\Delta T^i_j|$ for i is significantly smaller than $|\Delta T_j|$ and we suppose that ΔT^i_j is almost constant for all i, from equation (21) we have

$$\Delta T_j = \frac{1}{d} \sum_{i=1}^{d} G_j^i \Delta s \cdot \tag{23}$$

Equation (23) is an observation equation of the stochastic tube for the j-th reference ray. Then for application of the stochastic tube for traveltime tomography, the operator $(1/d)\sum G^i_j$ for G_j would be applied to the conventional observation equation (22) that uses only the reference ray. This application would make traveltime tomography possible, even if the structure includes small random heterogeneities characterized by an ACF with the statistic parameters. In actual cases, if the background velocity structure and the statistic properties of dominant heterogeneities are known, then we can calculate the kernel of the stochastic tube by using the estimated diffusion coefficient in advance (e.g. Fig.4).

5. Conclusions

For high frequency P-wave propagation, we proposed a stochastic tube tracing method, which could stochastically obtain the seismic wave paths passing through a three-dimensional random heterogeneous medium. The medium was constructed by a background structure and random perturbations. The ray obtained in the background structure was treated as a reference ray, and then seismic wave paths, of the same type as the reference ray, were modified by perturbations (diffusion) about the reference ray path in the medium. We suggested a stochastic approach to obtain such diffused paths as stochastic paths by using Brownian motion. The stochastic paths are determined by the traveltimes calculated along the paths having equal to or smaller than that of the reference ray path. The bundle of such stochastic paths was the stochastic tube, which was obtained by deriving and solving a stochastic differential equation. The stochastic tube represents the probable locations of the seismic wave paths of the same type as the reference ray in the random heterogeneous medium, where the heterogeneities are statistically characterized. We showed some numerical examples of the stochastic tubes for the first arrival P-wave in random heterogeneous media. As expected, high probabilities were observed around the reference ray path. The tubular width became large with decreasing anomaly size a and increasing velocity perturbation ε of the heterogeneities.

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Appendix

In the stochastic tube formulation, the stochastic differential equation (13) gives the unique solution (16), because the diffusion coefficient α and the drift coefficient b are assumed to satisfy the Lipschitz condition

$$\|\alpha(x,t)-\alpha(y,t)\| + \|b(x,t)-b(y,t)\| \leq K\|x-y\| \qquad \text{(A.1)}$$

for $0 \leq t < T$, arbitrary points x and y in the cross-section of the reference ray path, and a finite real number K. $\alpha = \alpha(x,t)$ and $b = b(x,t)$ are the vector and the time signatures. It is physically reasonable that the diffusion coefficient always be finite, inducing the first term on the left-hand side to also be finite. The second term on the left-hand side is simply replaced with $\|x-y\|/(T-t)$ from equation (15), because the first and the second terms in parenthesis on the right-hand side in equation (15) take on common values, irrespective of any x and y in a cross-section at time t of the reference ray. These two terms satisfy equation (A.1), which guarantees the uniqueness of the present solution.

確率的手法によるランダム不均質媒質中の地震波伝播経路の推定

宮澤理稔

要　旨

三次元ランダム不均質構造において二点を結ぶ地震波伝播経路を求める手法を，方程式を導出し解析的に解いて数値計算例を示すことで提案する。媒質が統計的関数によって特徴付けられるランダム不均質性を有する時，決定論的に地震波伝播経路を求めることが出来ないため，確率的に基準波線の周囲に確率的チューブを定義する。確率的チューブは震源と観測点を結び，基準波線と同じ波の伝播経路から成り，伝播経路として可能性のある多重経路を表現する。確率的チューブを表現するために，確率微分方程式を導出し解析的に解く。こうして求められた確率的チューブは地震波伝播経路の確率密度関数を示す。三次元ランダム不均質媒質中における初動Ｐ波に対する確率的チューブの例を，確率演算を伴う数値計算により求めて示す。

キーワード: 確率的チューブ，ランダム不均質，波線追跡，走時トモグラフィ，ブラウン運動