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Optimal Maintenance of Infrastructures under Natural Disaster Risk

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Synopsis

This study proposes a framework for dynamic evaluation and management of infrastructures facing natural disaster risk, especially, earthquakes. In this framework, the optimal maintenance problem is formulated as a stochastic impulse control problem, of which optimality condition is represented as a variational inequality problem (VIP). Our analyses reveal that the VIP reduces to a standard-form linear complementarity problem (LCP). This enables us to develop an efficient algorithm exploiting the recent advances in the theory of complementarity problem.

Keywords : non-steady-state Poisson jump; Dynamic Stochastic Control; Linear Complementarity Problem

1 Introduction

Recent earthquake disasters have repeatedly demonstrated the seismic vulnerability of grand-scale infrastructures, especially road, bridge and so on. While much attention has been paid on the preventive maintenance of these infrastructures, there are few studies which deal with natural disaster risk.

In the field of operations research, there are enormous numbers of studies which dealt with optimal maintenance problems of a deteriorating system (see Wang[6]). Stadje and Zuckerman [5] argues an optimal problem with endgenous repair degree. In the field of civil engineering, Rioja[4] discussed a problem of maintenance versus new investment in public infrastructures. Kobayashi and Ueda[3] formulated the optimal maintenance problem as a stochastic impulse control problem. They further analyse the problem by using the Markov decision process technique.

Althogh these studies have developed quite convenient methodology for the optimal maintenance problem, there are few studies which deal with natural disaster risk regarding several important facts in such a large-scale infrastructure management. Specifically, a) several natural disasters, in particular earthquakes, have non-steady-state behavior; b) there are financial constraints on maintenance cost whereby preventive maintenance takes a certain amount of time. Despite of their importance, these features have been neglected for either simplicity or computability in the past literature.

The purpose of this article is to propose a prototype of quantitative method for preventive maintenance problem of a infrastructure facing natural disaster risk. The most significant aspects of our method are both in a continuous time-space framework, and regarding above two features of the infrastructure management explicitly.

This paper is organized as follows. In Section 2, we formulate the optimal maintenance problem as stochastic control problem in a continuous time-state framework, where the optimality condition is represented as a Variational Inequality Problem (VIP). Section 3 sketches a numerical method for the problem exploiting the recent advances in the theory of complementarity problem. For this reason, we show the optimality condition reduces to a Linear Complementarity Problem (LCP) via certain variable transformation techniques. This numerical method is applied in Section 5. Section 6 concludes.

2 The Model

Consider one facility (e.g. one bridge in private road) managed by one administrator who intends to maximize the revenue during a certain period $[0, T]$. One can imagine the administrator holds a maintenance project as a PFI(*Private Finance Initiative*) contract of which mature is T.

2.1 Formulation

We denote the state of the facility as an aggregate one-dimensional variable. This variable called the functional level, of which value at time t is denoted as $P(t)$. We assume that the functional level follows a stochastic differential equation,

$$\begin{aligned} \mathrm{d}P(t) &= x(t)\,\mathrm{d}t - \mu(t,P)\,\mathrm{d}t \\ &\qquad + \sigma(t,P)\,\mathrm{d}W(t) - \eta(t,P)\,\mathrm{d}q(t), \quad (1) \\ P(0) &= P_0, \quad (2) \end{aligned}$$

where μ, σ, η are given functions of (t, P), and the initial value P_0 is a given constant. The dynamics (1) consists of four terms. The first term represents the enhancement of the functional level by the repair of the facility. We denote the increment of the functional level by $x(t)$ called the maintenance flow. per unit time. The second term, $\mu\,\mathrm{d}t$, states that the expected depreciation. The third term $\sigma\,\mathrm{d}W(t)$ adds a stochastic element of the depreciation in which $\mathrm{d}W(t)$ is the increment of a standardized Wiener-process $W(t)$ and the coefficient $\sigma(\cdot)$ denoted the instantaneous standard deviation of the functional level perturbation. The last term, $\eta(t,P)\,\mathrm{d}q(t)$ represents the discontinuous downward jump caused by the disaster, say an earthquake. While $\mathrm{d}q(t)$ is the increment of a non-stead-state Poisson process whose intensity at time t is denoted by a given function $\lambda(t)$, the coefficient $\eta(\cdot)$ represents the fragility of the functional level against the disaster.

Suppose that the revenues from the facility consist of three elements. The first one is the instantaneous revenue from the facility, of which value at time t is represented as a given function $\pi(t, P(t))$ [*1]. The second element is the maintenance cost which assumed the following linear function of the maintenance flow

$$C(t, x(t)) \equiv Ax(t), \quad (3)$$

where A is a given constant. The last element is the terminal payoff earned only at the expire date of the contract T. We denote the terminal payoff by a given function $F(T) \equiv F(P(T))$.

Suppose that there is a financial constraint on the maintenance cost. Specifically, we assume that the instantaneous maintenance cost has an upper limit $\bar{C}$. Therefore, the maintenance strategy at each moment $x(t)$ should satisfy the following constraint.

$$0 \le x(t) \le \bar{k} \equiv \bar{C}/A, \quad \forall t \in [0, T]. \quad (4)$$

The administrator decides the maintenance strategy $\{x(t)\}_{t=0}^{T}$ in order to maximize his own revenue from the facility during the contract period $[0, T]$. This profit maximization problem is formulated as the following stochastic control problem.

$$[\mathsf{P}] \quad \max_{\{x(t)\}_{t=0}^{T}} \mathcal{J}(0, P_0, x(\cdot)), \ \text{ s.t. (4)},$$

where ρ is a given constant which represents the discount rate of the cash flow stream, and $\mathcal{J}(t, P)$ is defined as follows:

$$\begin{aligned} &\mathcal{J}(t, P, x(\cdot)) \equiv \\ &\quad \mathrm{E}\Bigg[\int_t^T e^{-\rho(s-t)} \{\pi(s, P(s)) - C(s, x(s))\}\,\mathrm{d}t \\ &\qquad + e^{-r(T-t)} F(P(T)) \Bigg| P(t) = P\Bigg]. \quad (5) \end{aligned}$$

Eq.(5) represents the net present value of the cash flow stream during $[t, T]$ under the maintenance strategy $\{x(s)\}_{s=t}^{T}$.

2.2 Optimality Condition represented as a VIP

We define the value function of the problem [P], under a situation in which the functional level $P(t) = P$ is observed at time t, as follows:

$$\begin{aligned} V(t, P) \equiv \max_{\{x(s)\}_{s=t}^{T}} \mathcal{J}(t, P, x(\cdot)), \ \text{ s.t. (4)}, \\ \forall (t, P) \in [0, T) \times \mathcal{R}_+, \quad (6) \end{aligned}$$

where the terminal condition is

$$V(T, P) \equiv F(P), \quad \forall P \in \mathcal{R}_+. \quad (7)$$

Applying the DP (*Dynamic Programming*) principle, we obtain the following HJB (*Hamilton-Jacobi-Bellman*) equation

$$\begin{aligned} V(t, P) &= \max_{x(t)} \pi(t, P) - C(t, x(t)) \\ &\quad + e^{-\rho\,\mathrm{d}t} \mathrm{E}\left[V(t, P) + \mathrm{d}V(t, P) | P(t) = P\right], \\ &\qquad\qquad \text{s.t. (4)}. \quad (8) \end{aligned}$$

[*1] We can suppose the administrator intends to maximize the social benefit from the facility. In that case, the instantaneous revenue should be redefined as the 'instantaneous benefit', correspondingly.

That is, the value function $V(t, P)$ describes the maximum expected present value of revenues given the current functional level is P. Using the Ito lemma, the HJB equation (8) reduces to

$$\max_{x(t)} \pi(t,P) - C(t,x(t)) + x(t)V_P(t,P) + \mathcal{L}V(t,P) = 0, \tag{9}$$

where $\mathcal{L}$ is a differential operator defined as follows:

$$\begin{aligned}&\mathcal{L}V(t,P) \equiv \\ &\left\{\frac{\partial}{\partial t} - \mu(t,P)\frac{\partial}{\partial P} + \frac{1}{2}\sigma^2(t,P)\frac{\partial^2}{\partial P^2} - (\rho + \lambda(t))\right\} V(t,P) \\ &\qquad + \lambda(t)V[t, P - \eta(t,P)].\end{aligned} \tag{10}$$

Because there are no adjustment cost or costs associated with changing the level of maintenance, the problem [P] has a 'bang-bang' solution: the instantaneous maintenance level will be

$$x^*(t,P) = \begin{cases} 0 & \text{if } V_P(t,P) < A \\ \bar{k} & \text{if } V_P(t,P) \geq A \end{cases}. \tag{11}$$

Substituting the above optimal strategy into the HJB equation (9), we obtain the following two differential equation.

$$\pi(t,P) + \mathcal{L}V(t,P) = 0, \quad (x^*(t,P) = 0) \tag{12}$$

$$\pi(t,P) - \bar{C} + \mathcal{L}_1 V(t,P) = 0, \quad (x^*(t,P) = \bar{k}), \tag{13}$$

where

$$\mathcal{L}_1 \equiv \mathcal{L} + \bar{k}\frac{\partial}{\partial P}. \tag{14}$$

Since one of the two strategies 0 or $\bar{k}$ must be optimal, either (12) or (13) holds as equality. Hence, $V(t) \equiv \{V(t,P) | \forall P \in \mathcal{R}_+\}$ is the solution to the following VIP (*Variational Inequality Problem*) held at time t.

[VIP(t)]

$$\min.\left\{-\pi(t,P) - \mathcal{L}V(t,P),\ -\pi(t,P) + \bar{C} - \mathcal{L}_1 V(t,P)\right\} = 0, \quad \forall P \in \mathcal{R}_+.$$

where the terminal condition is Eq.(7).

3 Numerical Methods

Since the optimal maintenance problem [VIP(t)] has no analytical solution, we argue a numerical method in this section. First we reformulate the problem [VIP(t)] in a discrete framework. We then show the reformulated VIP reduces to a standard form LCP (*Linear Complementarity Problem*) via certain variable transformation techniques, whereby we develop an efficient algorithm exploiting the recent advances in the theory of complementarity problem.

3.1 Discretization

Let us consider a discrete grid in the time-state space with increments Δt and ΔP, and let $^{i,j} \equiv f(i\Delta t, j\Delta P)$ denote an arbitrary function at the grid points, where the indices $i \in \{0, 1, \cdots, \mathrm{I}\}$ and $j \in \{1, \cdots, \mathrm{J}\}$ characterize the location of the point with respect to state and time, respectively. In this setting, the [VIP(t)] is reformulated as follows:

$$[\mathrm{VIP}^i] \quad \min.\left\{-\boldsymbol{\pi}^i - \boldsymbol{L}^i\boldsymbol{V}^i - \boldsymbol{M}^i\boldsymbol{V}^{i+1},\ -\boldsymbol{\pi}^i + \bar{C}\mathbf{1} - \boldsymbol{L}_1^i\boldsymbol{V}^i - \boldsymbol{M}_1^i\boldsymbol{V}^{i+1}\right\} = \mathbf{0}, \tag{15}$$

where $\boldsymbol{V}^i \equiv \{V^{i,1}, \cdots, V^{i,\mathrm{J}}\}$, $\boldsymbol{\pi}^i \equiv \{\pi^{i,1}, \cdots, \pi^{i,\mathrm{J}}\}$ are given J-dimensional vectors, and $\mathbf{0}, \mathbf{1}$ are J-dimensional vectors with every elements $0, 1$, respectively. $\boldsymbol{L}^i, \boldsymbol{M}^i, \boldsymbol{L}_1^i, \boldsymbol{M}_1^i$ are $\mathrm{J} \times \mathrm{J}$ matrix obtained by approximating of the differential operators $\mathcal{L}, \mathcal{L}_1$ in a certain finite difference scheme. (See Appendix A). Correspondingly, the terminal condition (7) reduces to

$$\boldsymbol{V}^{\mathrm{I}} = \boldsymbol{F}, \tag{16}$$

where $\boldsymbol{F} \equiv \{F^1, \cdots, F^{\mathrm{J}}\}$.

Note that the [VIP^i] can be treated as an independent problem when the next period's value function $\boldsymbol{V}^{i+1}$ is known. Therefore, we can solve the [VIP^i] in a successive manner as follows: First, the value function at the expire date $\boldsymbol{V}^{\mathrm{I}}$ is known as the terminal condition (16); Then the unknown variable $\boldsymbol{V}^{\mathrm{I}-1}$ at the $\mathrm{I}-1$th period can be obtained as the solution of the [VIP$^{\mathrm{I}-1}$], in which $\boldsymbol{V}^{\mathrm{I}}$ is a known variable. Repeating this procedure successively, we can obtain whole unknown variables $\{\boldsymbol{V}^i | i = 0, 1, \cdots, \mathrm{I}-1\}$.

3.2 Reduction to A Standard Form LCP

The optimal maintenance problem [VIP^i] is not easy to handle because of its non-standard form. Consequently, we show that the [VIP^i] can reduces to a standardized form LCP (*Linear Complementarity Problem*) via a certain function transformation, whereby an efficient algorithm for calculating the problem [P] as shown in the Section 3.3.

First, let us consider the following new unknown variable.

$$\boldsymbol{X}^i \equiv -\boldsymbol{\pi}^i - \boldsymbol{L}^i\boldsymbol{V}^i - \boldsymbol{M}^i\boldsymbol{V}^{i+1}, \quad i = 0, 1, \cdots, \mathrm{I}-1. \tag{17}$$

If $\boldsymbol{L}^i$ is nonsingular, we can denote the $\boldsymbol{V}^i$ as the following linear transformation of $\boldsymbol{X}^i$.

$$\boldsymbol{V}^i \equiv -\left(\boldsymbol{L}^i\right)^{-1}\boldsymbol{X}^i - \boldsymbol{g}^i, \tag{18}$$

where

$$\boldsymbol{g}^i \equiv -\left(\boldsymbol{L}^i\right)^{-1}\left[\boldsymbol{\pi}^i + \boldsymbol{M}^i\boldsymbol{V}^{i+1}\right] \tag{19}$$

is a known variable at the ith period.

Substituting the variable transformations (17), (18) into [VIP^i], we obtain the following standard form LCP.

$$[\mathsf{LCP}^i] \quad \boldsymbol{X}^i \cdot \boldsymbol{G}^i(\boldsymbol{X}^i) = 0,\ \boldsymbol{X}^i \geq \boldsymbol{0},\ \boldsymbol{G}^i(\boldsymbol{X}^i) \geq \boldsymbol{0},$$

where $\boldsymbol{G}^i(\cdot)$ is a linear function of $\boldsymbol{X}^i$ defined as follows:

$$\boldsymbol{G}^i(\boldsymbol{X}^i) \equiv +\boldsymbol{L}_1^i \left(\boldsymbol{L}^i\right)^{-1} \boldsymbol{X}^i - \boldsymbol{h}^i, \tag{20}$$

and

$$\boldsymbol{h}^i \equiv \boldsymbol{\pi}^i - \bar{C}\boldsymbol{1} - \boldsymbol{L}_1^i \boldsymbol{g}^i - \boldsymbol{M}_1^i \boldsymbol{V}^{i+1} \tag{21}$$

is a known variable at the ith period.

3.3 Algorithm

The key of the above sections are twofold: the [VIP^i] reduces to standard form LCP; the [VIP^i] can be solved in a successive manner. Thus, an efficient algorithm the maintenance problem [VIP^i] can be denoted as follows: (a) Compute $\boldsymbol{X}^i$ as the solution of the [LCP^i], regarding $\boldsymbol{V}^{i+1}$ as a constant; (b) Obtain $\boldsymbol{V}^i$ via the variable transformation (18); (c) Repeat these two steps by turns.

While there are a large variety of numerical algorithms for the standard form LCP such as [LCP^i], we adopt the *merit function approach* that appeared with the recent advances in the theory of operations research. Generally, this merit function approach is quite effective and does not require strict conditions for convergence compared to the traditional methods, for instance, diagonalization method, Lemke method, projection method, and so on (See Ferris and Pang[1]).

This approach reduces the [LCP^i] to an convex problem with one-dimensional objective function $\Phi^i(\boldsymbol{X}^i)$ called the *merit function*. The merit function is a continuous $\mathcal{R}^J \to \mathcal{R}_+$ mapping whose value satisfies that $\Phi^i(\boldsymbol{X}^i) = 0$ when $\boldsymbol{X}^i$ is the solution of the [LCP^i] and $\Phi^i(\boldsymbol{X}^i) > 0$ when $\boldsymbol{X}^i$ is not the solution. In this study, we introduce the *Fukushima merit function*[2] defined as follows:

$$\Phi^i(\boldsymbol{X}^i) \equiv -\boldsymbol{G}^i(\boldsymbol{X}^i) \cdot \boldsymbol{H}^i(\boldsymbol{X}^i) - \frac{1}{2}\boldsymbol{H}^i(\boldsymbol{X}^i) \cdot \boldsymbol{H}^i(\boldsymbol{X}^i), \tag{22}$$

where

$$\boldsymbol{H}^i(\boldsymbol{X}^i) \equiv \left[\boldsymbol{X}^i - \boldsymbol{G}^i(\boldsymbol{X}^i)\right]_+ - \boldsymbol{X}^i, \tag{23}$$

and $[\boldsymbol{Z}]_+$ is an operator of projection on the positive real space $\mathcal{R}_+^J$, of which kth element is defined as $\max.\left\{Z^k, 0\right\}$.

An efficient algorithm for [LCP^i] using Fukushima merit function (22) can be described as the following procedure[2], which requires $\boldsymbol{V}^{i+1}$ previously obtained.

In summary, whole of the numerical method for the optimal maintenance problem [P] is described as follows:

Algorithm 1 Fukushima Procedure

procedure FUKUSHIMA LCP($i, \boldsymbol{V}^{i+1}$)
  $\boldsymbol{X}^{i(1)} \in \mathcal{R}_+^J$ ▷ initial feasible solution
  $n := 1$
  repeat
    $\boldsymbol{d}^n := \boldsymbol{H}^i(\boldsymbol{X}^i)$ ▷ descent direction
    Find α such that
      $\min_{\alpha \in [0,1]} \Phi^i(\boldsymbol{X}^{i(n)} + \alpha \boldsymbol{d}^{(n)})$ ▷ line search
    $\boldsymbol{X}^{i(n+1)} := \boldsymbol{X}^{i(n)} + \alpha \boldsymbol{d}^{(n)}$ ▷ iteration
    $n := n + 1$
  until $\boldsymbol{X}^{i(n)}$ converges to the solution
  return $\boldsymbol{X}^{i(n)}$.
end procedure

Algorithm 2 Main Algorithm

$\boldsymbol{V}^I = \boldsymbol{F}$ ▷ terminal condition
for $i := I - 1$ to $i = 0$ step -1 **do**
  Compute $\boldsymbol{X}^i$ by using the FUKUSHIMA LCP($i, \boldsymbol{V}^{i+1}$) procedure.
  Obtain $\boldsymbol{V}^i$ by substituting $\boldsymbol{X}^i$ into Eq. (18)
end for

4 Example

We finally apply our method to a simple example and show several numerical results. The purpose of this section is to show an illustrative example of our framework and to confirm the algorithm solves the problem. For this reason, we adopt several restriction on the model in this section. Remember these assumption *do not* affect the essence of our method.

First, let us assume that the dynamics of the functional level as the following geometric form:

$$\frac{\mathrm{d}P(t)}{P(t)} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) + \eta\,\mathrm{d}q(t). \tag{24}$$

We further suppose a steady-state Poisson process for the catastrophic event whose intensity is denoted $\lambda \equiv \lambda(t)$[*2].1

In the rest of this section, we show the value function and the optimal strategy under the following parameters:

$$T = 50, \quad \mu = 0.02, \quad \sigma = 0.01, \quad \rho = 0.1 \tag{25}$$
$$\lambda = \frac{1}{50}, \quad \eta = 0.5, \quad \bar{C} = 0.5, \quad A = 1.0,$$

and the linear instantaneous revenue

$$\pi(t, P) \equiv 0.1P - 0.6. \tag{26}$$

[*2] Note the intensity is a reciprocal number of the recurrence interval of the catastrophic events.

4.1 Value Function

Fig.1 displays the value function at each point in the time-state space. While the two horizontal axes represents time and the state variable (i.e. the functional level), the vertical axes represents the value function. This figure shows that the value function is increasing respect to the functional level at arbitrary moment.

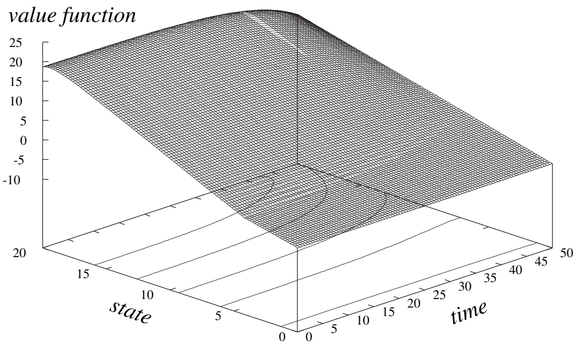

Fig. 1 Value Function

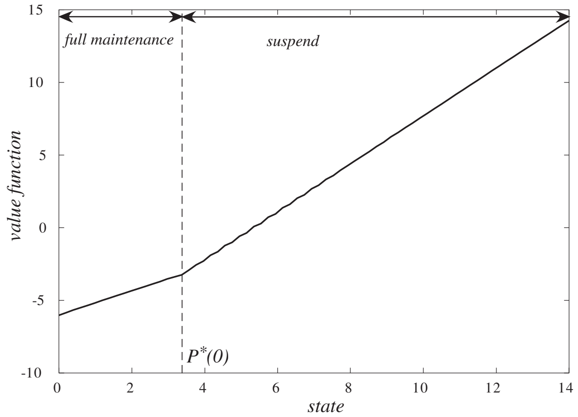

Fig. 2 Value Function at $t = 0$

Fig.2 shows a slice of Fig.1 at the initial time $t = 0$. In this figure, the horizontal axis represents the functional level and the vertical one denotes the value function, respectively. This figure shows that the value function consists of two functions connected at the level $P^*(0)$, at where the optimal strategy switches from 'full maintenance' to 'suspend'. In other words, if the functional level is below $P^*(0)$, the administrator maintain the facility as much as possible. Contrarily, the administrator does nothing but wait for a moment in the case of the functional level exceeds $P^*(0)$.

4.2 Maintenance-Suspend Threshold

The switching threshold at every moment $P^*(t)$ is displayed in Fig.3, of which horizontal axis represents time and the vertical axis is the functional level. In this figure, three thresholds are shown respectively the three case of the catastrophic intensity, $\lambda = \frac{1}{20}, \frac{1}{100}$ and $\frac{1}{\infty}(= 0)$. The last case represents a situation of no disaster risk exists. Fig. 3 shows that as the catastrophic intensity increases (i.e. the reccurence interval decreases), the threshold shifts upward (i.e. the administrator intends to 'fully maintain' frequently).

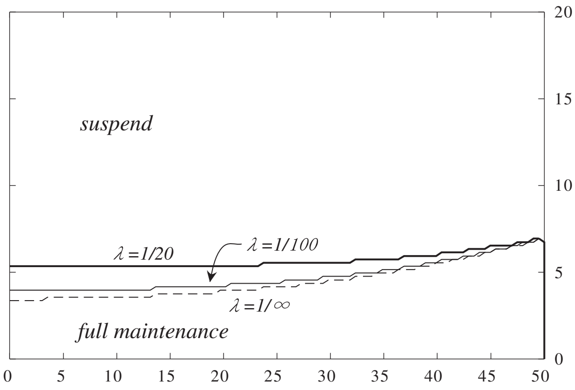

Fig. 3 Optimal Maintenance Strategy

5 Conclusion

This study proposed a prototype framework for quantitative analysis on an optimal maintenance problem of a certain infrastructure faces catastrophic risk. In this framework, we have shown that the finite difference method is still convenient to deal with non-steady-state Poisson processes. Further we have revealed not only a 'bang-bang' strategy is optimal under a linear maintenance cost, but also the optimality condition can be formulated as a variational inequality problem (VIP). We finally have shown the VIP reduces a linear complementarity problem by using certain variable transformation techniques. This make us enable to develop an efficient algorithm exploiting the recent advances in the complementarity problem theory.

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Appendix A The Finite Difference Method

We approximate the partial derivatives in the [VIP(t)] as follows

$$\frac{\partial V(t,P)}{\partial t} \approx \frac{V^{i+1,j} - V^{i,j}}{\Delta t},$$
$$\frac{\partial V(t,P)}{\partial P} \approx \beta \frac{V^{i+1,j+1} - V^{i+1,j-1}}{2\Delta P} + (1-\beta)\frac{V^{i,j+1} - V^{i,j}}{2\Delta P}$$
$$\frac{\partial^2 V(t,P)}{\partial P^2} \approx \beta \frac{V^{i+1,j+1} - 2V^{i+1,j} + V^{i+1,j-1}}{(\Delta P)^2} + (1-\beta)\frac{V^{i,j+1} - 2V^{i,j} + V^{i,j-1}}{(\Delta P)^2},$$

By applying the above finite difference scheme, $\mathcal{L}V(t,P)$ is rewritten as follows:

$$\mathcal{L}V(t^i,P) \approx \boldsymbol{L}^i \boldsymbol{V}^i + \boldsymbol{M}^i \boldsymbol{V}^{i+1} + \lambda^i \boldsymbol{D}^i \boldsymbol{V}^i, \qquad (27)$$

where $\boldsymbol{L}^i$ is a trigonal matrix defined as follows:

$$\boldsymbol{L}^i \equiv \begin{bmatrix} b^{i,1} & c^{i,1} & 0 & \cdots & 0 & 0 & 0 \\ a^{i,2} & b^{i,2} & c^{i,2} & \cdots & 0 & 0 & 0 \\ 0 & a^{i,3} & b^{i,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b^{i,\mathrm{J}-2} & c^{i,\mathrm{J}-2} & 0 \\ 0 & 0 & 0 & \cdots & a^{i,\mathrm{J}-1} & b^{i,\mathrm{J}-1} & c^{\mathrm{J}-1} \\ 0 & 0 & 0 & \cdots & 0 & a^{i,\mathrm{J}} & b^{i,\mathrm{J}} \end{bmatrix},$$

and $\boldsymbol{M}^i$ is a trigonal matrix, of which diagonal elements $b^{i,j}$ is replaced by $d^{i,j}$. For instance, when we adopt the Crank-Nicholson scheme (i.e., $\beta = 1/2$), these elements are $a^{i,j} \equiv \frac{\mu^{i,j}/\Delta P + (\sigma^{i,j}/\Delta P)^2}{4}$, $b^{i,j} \equiv -\frac{1}{\Delta t} - \frac{(\sigma^{i,j}/\Delta P)^2}{2} - (r + \lambda^i)$, $c^{i,j} \equiv \frac{-\mu^{i,j}/\Delta P + (\sigma^{i,j}/\Delta P)^2}{4}$, and $d^{i,j} \equiv \frac{1}{\Delta t} - \frac{(\sigma^{i,j}/\Delta P)^2}{2}$, respectively. In this approximation, $\boldsymbol{D}^i$ is a lower triangular matrix, of which (j,k) element is defined as follows:

$$\delta^{i,j,k} \equiv \begin{cases} 1 & \text{if } k \leq \frac{\eta(t^i,P^j)}{\Delta P} < k+1 \\ 0 & \text{otherwise} \end{cases} \qquad (28)$$

Correspondingly, $\mathcal{L}_1 V(t,P)$ is approximated as follows:

$$\mathcal{L}_1 V(t^i,P) \approx \boldsymbol{L}_1^i \boldsymbol{V}^i + \boldsymbol{M}_1^i \boldsymbol{V}^{i+1} + \lambda^i \boldsymbol{D}^i \boldsymbol{V}^i, \qquad (29)$$

where $\boldsymbol{L}_1^i, \boldsymbol{M}_1^i$ are tridiagonal matrices as same as $\boldsymbol{L}^i, \boldsymbol{M}^i$ but their elements $a^{i,j}, c^{i,j}$ are replaced by $e^{i,j} \equiv \frac{(-\bar{k}+\mu^{i,j})/\Delta P + (\sigma^{i,j}/\Delta P)^2}{4}$ and $f^{i,j} \equiv \frac{(\bar{k}-\mu^{i,j})/\Delta P + (\sigma^{i,j}/\Delta P)^2}{4}$, under the Crank-Nicholson scheme.

自然災害リスク下での施設の最適補修戦略

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要旨

本研究の目的は，災害リスクに直面する社会基盤施設の最適補修問題に対する定量的分析手法のプロトタイプの提案である．具体的には，まず，最適補修問題を確率的インパルス制御問題として定式化し，その最適性条件が変分不等式問題 (VIP: *Variational Inequality Approach*) として記述されることを示す．次に，この VIP が，適当な関数変換によって，標準形の線形相補性問題 (LCP: *Linear Complementarity Problem*) に帰着することを明らかにする．この分析結果に基づき，最近の相補性問題理論を活用した効率的解法を開発する．

キーワード： 予防的保全，非定常 Poisson 過程，確率的インパルス制御問題

COE-33

災害リスク下での社会基盤施設の最適補修戦略

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1　はじめに

道路や橋梁などの社会基盤施設（以下，施設）の多くは，一般に，これらの施設の新規建設には多くの費用を必要とし，その供用期間は数十年以上にも及ぶ．そのため，これらの社会基盤施設に対しては，長期的な補修を明示的に考慮した性能設計が必要不可欠である．こうした社会基盤施設の補修戦略および性能設計を行うには，以下の 3 つの特性を考慮する必要がある：a）地震などの予測不可能な自然災害によって構造物が破壊され，施設からのサービス水準が著しく劣化するリスク（災害リスク）が存在する；b）施設の自然劣化による減耗が非定常的な動学的不確実性をもつ；c）各年度の補修費用に上限が存在する．本研究では，これらの要因を全て考慮した枠組の下で，最適補修問題に対する定量的分析手法の提案を目的とする．

2　定式化

本研究では，社会基盤施設として高速道路における橋梁や高架といった単一の構造物を対象とする．そして，有限の管理期間 $t \in [0,T)$ に施設から得られるサービス水準に影響する機能水準 e.g. 路面性状）を，一次元変数 $P(t)$ で集約的に表現できるとし，以下の確率過程で表現する．

$$\mathrm{d}P(t) = x(t)\,\mathrm{d}t - \mu(t,P)\,\mathrm{d}t + \sigma(t,P)\,\mathrm{d}W(t) - \eta(t,P)\,\mathrm{d}q(t), \quad P(0) = \bar{P}. \quad (1)$$

この式の右辺第 1 項は補修 $x(t)$ による機能水準の向上を，第 2, 3 項は自然劣化による減耗のドリフトおよびボラティリティを，第 4 項は自然災害による不連続的な劣化を表す．ここで，$q(t)$ は強度 $\lambda(t)$ の非定常 Poisson 過程であり，η は t, P についての既知関数とする．

状態 (t,P) において補修量 x が選択されているとき，この施設から単位時間当たりに発生する純便益を，$F(t,P) \equiv \pi(t,P) - C(x)$ とする．この式の右辺は，それぞれ，利用者（租）便益および補修費用を表す．ここで，補修費用は補修速度に対して線形 (i.e. $C(x) = Ax$ とし，任意の時刻において上限 $\bar{C}$ が設けられているとする (i.e. $C(x) \equiv Ax < \bar{C}$).

上述の枠組の下で，施設管理者は，供用期間 $[0,T)$ 中に獲得する便益の期待現在正味価値を最大化するように補修戦略 $\{x(t)|t \in [0,T)\}$ を決定する．この行動は，

$$[\mathrm{P}] \quad \max_{\{x(t)\in\}_0^T} \mathrm{E}\left\{ \int_0^T e^{-\rho t} \{\pi(t,P(t)) - C(x(t))\}\,\mathrm{d}t \,\middle|\, P(0) = \bar{P} \right\}.$$

と定式化される．

3　最適性条件

問題 [P] の最適値関数を以下のように定義する．

$$V(t,P) \equiv \max_{\{x(s)\in\}_t^T} \mathrm{E}\left\{ \int_t^T e^{-\rho(s-t)} \{\pi(s,P(s)) - C(x(s))\}\,\mathrm{d}s \,\middle|\, P(t) = P \right\}$$

この式に DP 原理を適用して整理すれば，時刻 t において成立するべき以下の HJB(*Hamilton-Jacobi-Bellman*) 方程式を得る．

$$\max_{0\le x\le k} \pi(t,P) + \mathcal{L}V(t,P) + x\{-A + V_P(t,P)\} = 0, \ \forall P \in \mathcal{R}_+. \quad (2)$$

ここで，$\mathcal{L}$ は偏微分作用素で，以下の式で定義される．

$$\mathcal{L}V(t,P) \equiv V_t - \mu V_P + \frac{1}{2}\sigma^2 V_{PP} + (\rho-\lambda)V + \lambda(t)V(t, P-\eta(t,P))$$

この作用素は，施設の性能水準 P の確率微分方程式 (1) から一意に決まる．HJB 方程式 (2) は，補修速度 x に関して線形であるため，最適補修戦略は $x = 0$ もしくは $x = K/A$ の Bang-Bang 制御となる．これより，HJB 方程式 (2) は以下の変分不等式問題 (VIP:*Variational Inequality Problem*)：

$$[\mathrm{VIP}(t)] \quad \min.\Big\{\pi(t,P) + \mathcal{L}V(t,P),\ \pi(t,P) + \bar{k}V_P + \mathcal{L}V(t,P) - \bar{C}\Big\} = 0, \quad \forall P \in \mathcal{R}_+.$$

として書き直せる．終端条件は，$V(T,P) = 0$ とする．

4　最適補修問題の解法

最適補修問題 [P] は，各時刻 t ごとに成立する変分不等式問題 [VIP(t)] に分解できることが判った．筆者らは，このような VIP が，適切な関数変換によって数理計画分野で良く知られる標準形の相補性問題に帰着することを明らかにしている [1]．この分析結果を用いれば，問題 [VIP(t)] に対しても，相補性問題の数値解法に関する最新の研究成果を活用した効率的計算法が開発できる．その詳細なアルゴリズムおよび計算結果については，発表会で報告する予定である．

参考文献

[1] 赤松隆, 長江剛志：不確実性下での社会基盤投資・運用問題に対する変分不等式アプローチ, 土木学会論文集, 2004, 投稿中.