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Computational Modeling of Large Deformation of Saturated Soils Using An ALE Finite Element Method

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Synopsis

An arbitrary Lagrangian Eulerian (ALE) finite element method is developed for large deformation problems of saturated soils. A fluidal-elasto-plastic constitutive equation is employed for the soils. Using an incremental approach, coupled ALE finite element formulations are derived. To an existing program source code written by updated Lagrangian scheme, the ALE formulation is implemented by using an operator-split technique. This operator-split algorithm is composed of a Lagrangian step and an Eulerian step. The Lagrangian step is a pure updated Lagrangian calculation. The Eulerian step is performed using mesh smoothing and data transferring schemes. The proposed method is illustrated by numerical simulation of responses of an embankment subjected an earthquake motion.

Keywords: saturated soils; large deformation; arbitrary Lagrangian Eulerian method; finite element method; liquefaction

1. Introduction

Many numerical analysis problems in geotechnical engineering (e.g., liquefaction induced large ground displacement, driving of a pile, and penetration problems) involve large deformation. Thus it is necessary to take into consideration the large strain of saturated soils. Usually the conventional updated Lagrangian finite element methods are applied to analyze nonlinear response of saturated soils. However, it is required to pay particular attention to the mesh characteristics. When extensive mesh distortion and elements entanglement arise, the elements may have large strain within the body, leads to a loss in accuracy, the Jacobian at some integration points may approach to zero and causes ill-conditioned stiffness matrices. In order to surmount these deficiencies, an arbitrary Lagrangian Eulerian (ALE) finite element method was adopted in this paper.

In finite element analysis of large deformation problem, two numerical formulations have been extensively used, the Lagrangian approach and the Eulerian approach.

In a Eulerian formulation, the relevant quantities are described with respect to the position in space coordinate, which is used to label a material particle in the continuum at present time. Using the Eulerian description, we can choose a fixed mesh, but it is difficult to convert and migrate the material particles on the fixed mesh. Therefore, it is not appropriate to present the free boundary condition and simulate the material deformation history for nonlinear solid mechanics.

In a Lagrangian formulation, relevant quantities are described with respect to the initial coordinate (Total Lagrangian) or fixed to the geometry at the beginning of the time step and moving with the material (Updated Lagrangian). When the finite element discretization is implemented, the configuration of the body is covered with a mesh. A node is then associated with the same material particle throughout the deformation process of the body. The mesh is then deformed along with the body. Lagrangian formulation is well suited for problems concerning path-dependent material with free surface conditions. If severe mesh distortion and elements entanglement occur in case of large deformation, the Lagrangian reference state would be unsuitable for further step analysis, and leads to less accurate results or even to an interruption of the calculation.

The ALE method combines the advantages of the two procedures described above while avoiding their drawbacks. The general theory of ALE formulation is based on choosing a reference configuration, which is independent of both the material and spatial ones. In an ALE analysis, the computational reference system (finite element mesh) is neither attached to the material nor fixed in space. The mesh is deformed as in Lagrangian formulation, but independently from the material body as in Eulerian formulation and keeping the mesh regularity. Consequently, ALE formulations can handle path-dependent material behavior and free surface condition while keeping the mesh regularity.

The ALE method was first developed in fluid mechanics. It has been successfully applied to fluid and structure coupling problems by Belytschko and Kennedy (1978), and then has been implemented in finite element analyses of a solid mechanics by Haber (1982), Hughes et al. (1981), Benson (1989), Liu et al. (1988, 1991) and Ghost et al. (1991). A general ALE finite element formulation in nonlinear solid mechanics has been proposed by Gadala and Wang (1997, 1998). However, a general ALE finite element formulation for porous media has not been well established.

The ALE procedures in the literature can be divided in coupled and operator-split ALE formulations. In the first formulation, the fully coupled Lagrangian-Eulerian equations involving both material and mesh velocities are solved (Liu et al., 1991; Gadala and Wang, 1998). In the second approach, an operator-split scheme is used and the coupled Lagrangian-Eulerian equations are split and solved separately (Benson, 1989; Aymone et al., 2001).

Different from nonlinear solid material, saturated soil is a two-phase material with a soil skeleton and a pore fluid phase. It is necessary to consider the two-phase interaction for ALE formulation. In this study, a fluidal-elasto-plastic constitutive model, which was proposed by Moon and Sato et al. (2000) for saturated soils, is adopted to simulate liquefaction and following ground flow. The coupled ALE formulation for fully saturated soils is derived on the basis of Biot's theory and an incremental approach. The implementation of the operator-split ALE method to simulate large deformation is also presented. Numerical simulation of an embankment subjected an earthquake motion is carried out to illustrate the proposed method.

2. Governing Equations

Saturated soil is a two-phase material with a soil skeleton and a pore fluid phase. The soil skeleton is compressible and may be deformed according to a non-linear constitutive criterion. The complete Biot equation governing deformable porous medium can be expressed as:

$$\sigma_{ij,j} + \rho b_i - \rho \dot{v}_i - \rho^f \dot{w}_i = 0 \tag{1}$$

where σ_{ij} is the Cauchy total stress in the combined solid and fluid mixture, b_i the body force acceleration, ρ^f the density of the pore water, ρ the apparent density of saturated soils, v_i the velocity of the soil skeleton and w_i the average relative velocity of seepage.

The pore fluid seepage flows through the pores according Darcy's law. The generalized Darcy equation can be written as:

$$w_i = k\left(-p_{,j} + \rho^f b_j - \rho^f \dot{v}_j - \rho^f \frac{\dot{w}_j}{n}\right) \tag{2}$$

where p is the pore pressure (taken positive when compressive), n the porosity and k the permeability.

The continuity equation can be expressed as (Oka et al., 1994; Shibata et al., 1991; Akai and Tamura, 1978):

$$w_{i,i} + v_{i,i} + \left(\frac{n}{K^f} + \frac{1-n}{K^s}\right)\dot{p} = 0 \tag{3}$$

where K^s is the bulk modulus of the solid material and K^f the bulk modulus of the fluid material.

If the following items are adopted when formulating the governing equations (Di and Sato, 2004),

[1] The Large strain is considered.

[2] The gradients of $\ln(n)$ and $\ln(\rho^f)$ are so small that their quadratic terms can be ignored and $\left(\frac{\partial^2(\ln n)}{\partial x_i^{\,2}}\right)_i = 0$.

[3] The u-p formulation is adopted. The acceleration of fluid phase related to the soil skeleton can be neglected (Zienkiewicz et al., 1980, 1984).

[4] Soil particles are incompressible.

Then the equilibrium equation of motion for total mixture of soil skeleton and fluid phase is simplified as:

$$\sigma_{ij,j} + \rho b_i - \rho \dot{v}_i = 0, \tag{4}$$

and the continuity equation can also be obtained as:

$$-\frac{k}{g}\dot{l}_{ii} - \frac{k}{\gamma^f} p_{E,ii} + l_{ii} + \frac{n}{K^f}\dot{p}_E = 0 \tag{5}$$

where p_E is the excess pore pressure and l_{ij} the symmetric rate of deformation tensor.

3. ALE Kinematics

For the motion and deformation of a body, the material particles are labeled by the coordinates, X_i, at their initial positions at time $t = 0$, the current positions of these particles are located by the coordinates, ${}^t x_i$, in the spatial domain at the time t. In the ALE description, a referential domain, which composed of the coordinates χ_i of grid points of mesh at the time t, is employed to describe state variables. Define that u_i and v_i are the displacement and velocity of the soil skeleton, $\hat{u}_i$ and $\hat{v}_i$ are the displacement and velocity of the mesh grid on the material. The soil skeleton displacement u_i and the mesh grid displacement $\hat{u}_i$ on the material have the following values:

$$u_i = {}^t x_i - X_i \tag{6}$$

$$\hat{u}_i = {}^t x_i - \chi_i \tag{7}$$

The material velocity v_i and the mesh velocity $\hat{v}_i$ can be obtained by differentiating the equations of material motion and mesh motion presented previously with respect to time while keeping the particle X_j or the mesh grid point χ_j fixed. A convective velocity c_i should be introduced to mapping the convective effects between the material and grid, and is denoted as:

$$c_i = v_i - \hat{v}_i \tag{8}$$

In the ALE formulation, the acceleration of mesh grid is not important. So only the material accelerations a_i and $\hat{a}_i$ are needed, it can be expressed respectively as:

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{X_j} \tag{9}$$

$$\hat{a}_i = \left.\frac{\partial v_i}{\partial t}\right|_{\chi_j} \tag{10}$$

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{{}^t x_j} + v_j \frac{\partial v_i}{\partial^t x_j} \tag{11}$$

$$a_i = \hat{a}_i + c_j \frac{\partial v_i}{\partial^t x_j}. \tag{12}$$

Taking a function f defined on the current configuration according to the spatial coordinates ${}^t x_j$, the f can be stress, strain or any state variables. The material, spatial and ALE computational referential time derivative of function f can be expressed as:

$$\dot{f} = \left.\frac{\partial f}{\partial t}\right|_{X_i} = \left.\frac{\partial f}{\partial t}\right|_{{}^t x_i} + v_j \frac{\partial f}{\partial^t x_j} \tag{13}$$

$$\overset{0}{f} = \left.\frac{\partial f}{\partial t}\right|_{{}^t x_i} \tag{14}$$

$$f' = \left.\frac{\partial f}{\partial t}\right|_{\chi_i} = \left.\frac{\partial f}{\partial t}\right|_{{}^t x_i} + \hat{v}_j \frac{\partial f}{\partial^t x_j} \tag{15}$$

From Eqs. (15) and (13), the referential derivative f' can be related to the material derivative $\dot{f}$ by:

$$\dot{f} = f' + c_j \frac{\partial f}{\partial^t x_j} \tag{16}$$

Let the volume dv in spatial configuration be the image of a volume dV in material configuration, we get:

$$dv = JdV \tag{17}$$

where $J = \det\left(\frac{\partial X_i}{\partial^t x_j}\right)$.

Using the mass density ρ of dv and the mass density ρ_0 of dV, the classical mass conservation equation is:

$$\rho = J^{-1}\rho_0 \tag{18}$$

The material derivative form of this equation gives:

$$\dot{\rho} = -\rho \frac{\partial v_i}{\partial^t x_i} \tag{19}$$

In the ALE formulation, the corresponding referential derivative of mass density can be obtained as:

$$\rho' = -\rho \frac{\partial v_i}{\partial^t x_i} - c_j \frac{\partial \rho}{\partial^t x_j} \tag{20}$$

4. Constitutive Relationships

A fluidal-elasto-plastic constitutive model has been proposed to model the nonlinear behavior of saturated soils (Moon, Sato and Uzuoka, 2000). The constitutive model combines an effective cyclic elasto-plastic model (Oka et al., 1999) with the Newtonian viscous fluid model, and uses a coefficient α to control phase changes. It is able to describe liquefaction of saturated soils and following ground flow. The α has been proposed by Moon et al. (2000) in the following form:

$$\alpha = ae^{\left(b\left(1-\frac{\sigma'_m}{\sigma'_{m0}}\right)\right)} \tag{21}$$

where σ'_m is mean effective stress, σ'_{m0} is the initial mean effective stress, a and b are the constant parameter for the material.

Since large deformation is considered, the Jaumann stress, which gives an objective measure of stress, is adopted here.

$$\dot{\sigma}^J_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\omega_{jk} - \sigma_{jk}\omega_{ik} \tag{22}$$

where $\dot{\sigma}_{ij}$ is the rate of stress, ω_{ij} is the skew symmetric spin tensor.

Then the fluidal-elasto-plastic constitutive model can be expressed in the present formulation:

$$\dot{\sigma}^J_{ij} = (1-\alpha)\dot{\sigma}^{ep}_{ij} + \alpha\dot{\sigma}^{vf}_{ij} - \dot{p}\delta_{ij} \tag{23}$$

where $\dot{\sigma}^J_{ij}$ is the Jaumann stress, $\dot{\sigma}^{ep}_{ij}$ is the stress which is evaluated by the effective cyclic elasto-plastic model (Oka et al., 1999), $\dot{\sigma}^{vf}_{ij}$ is the stress which can be obtained using the Newtonian viscous fluid model, δ_{ij} the Kronecker delta. We get:

$$\begin{aligned}\dot{\sigma}^J_{ij} &= (1-\alpha)D^{ep}_{ijkm}l_{km} \\ &\quad + \alpha\left(\lambda \dot{l}_{kk}\delta_{ij} + 2\mu \dot{l}_{ij}\right) - \dot{p}\delta_{ij}\end{aligned} \tag{24}$$

where μ and λ are the viscous efficients, D^{ep}_{ijkm} is the elasto-plastic tensor of the effective cyclic elasto-plastic constitutive model.

Using Eq. (16), the grid time derivative of the stress is written as:

$$\begin{aligned}\sigma'_{ij} &= \dot{\sigma}_{ij} - (v_k - \hat{v}_k)\frac{\partial \sigma_{ij}}{\partial x_k} \\ &= (1-\alpha)D^{ep}_{ijkm}l_{km} + \alpha\left(\lambda \dot{l}_{kk}\delta_{ij} + 2\mu \dot{l}_{ij}\right) \\ &\quad + \sigma_{ik}\omega_{jk} + \sigma_{jk}\omega_{ik} \\ &\quad - (v_k - \hat{v}_k)\frac{\partial \sigma_{ij}}{\partial x_k} - \dot{p}\delta\end{aligned} \tag{25}$$

Equation (25) shows that the grid time derivative of the stress is found using the Jaumann stress rate, and the fluidal-elasto-plastic constitutive model can be employed without further modification in the ALE finite element analysis.

5. The Coupled ALE Formulations

Using an incremental approach, coupled ALE finite element formulations for saturated soils are derived in this section.

Within an ALE analysis, the finite element mesh is neither attached to the material nor fixed in space. It has a motion, which is independent of the material. Two sets of coordinate systems have to be defined. As shown in Fig. 1, one coordinate system is attached to a material point, ${}^tP^M$, and moving to ${}^{t+\Delta t}P^{M}$ with material deformation, another is corresponded to a computational referential grid, ${}^{t}P^{G}$, which moves to ${}^{t+\Delta t}P^{G}$ independently according to a user-defined mesh motion. In an incremental approach analysis, it is assumed that the configuration at time t is known on the material domain, which is equal to the computational referential domain. In a Lagrangian formulation the computational domain is equal to the material domain at the configuration of time $t+\Delta t$. In an ALE formulation the computational domain is formed by the displacement of all grid points, and not necessarily equal to the material domain at the configuration of time $t+\Delta t$. The deformation and the values of the state variables at time $t+\Delta t$ have to be calculated on the new computational domain. In this paper, left superscripts of a quantity indicate the configuration at which the quantity occurs. Left subscripts indicate the configuration with respect to which the quantity is measured, and may not used if it is same to the configuration at which the quantity occurs.

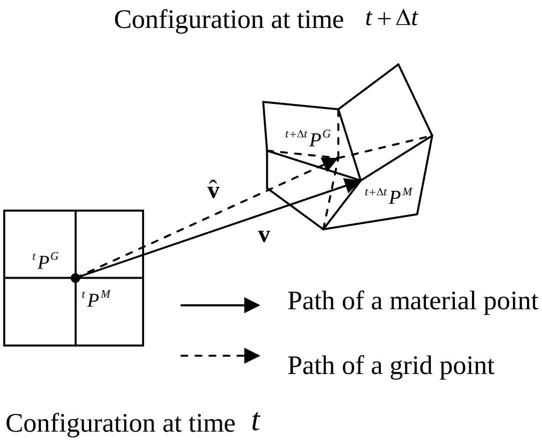

Fig. 1 Material point and mesh grid movements in ALE formulation

Although the mesh and material motions are independent from each other, there exists a one-to-one mapping between material and computational domains. The boundaries of the two domains should coincide, requiring that:

$$(v_i - \hat{v}_i)n_i = 0 \quad \text{on the boundary.} \tag{26}$$

where n_i is the normal vector at any point on the boundary.

Because the incremental approach is used, all quantities in the governing equations of saturated soils should be transformed into the known configuration at time t. During the transformation, each quantity should be related to the computational coordinate system χ_j. The ${}^{t+\Delta t}x_i$ are related to the coordinate ${}^{t}x_i$ through:

$${}^{t+\Delta t}x_i = {}^{t}x_i + \hat{u}_i \tag{27}$$

From Eq. (27), we can get:

$$\frac{\partial^{t+\Delta t}x_i}{\partial^{t}x_i} = 1 + \frac{\partial \hat{u}_i}{\partial^{t}x_i} \tag{28}$$

$$\frac{\partial^{t}x_i}{\partial^{t+\Delta t}x_i} = \frac{1}{1+\dfrac{\partial \hat{u}_i}{\partial^{t}x_i}} = \frac{1-\dfrac{\partial \hat{u}_i}{\partial^{t}x_i}}{1-\dfrac{\partial \hat{u}_m}{\partial^{t}x_m}\dfrac{\partial \hat{u}_n}{\partial^{t}x_n}} \approx 1 - \frac{\partial \hat{u}_i}{\partial^{t}x_i} \tag{29}$$

The symmetric rate of the deformation tensor ${}_{t+\Delta t}l_{ij}$ referred to the configuration at time $t+\Delta t$ is defined by:

$${}_{t+\Delta t}l_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial^{t+\Delta t}x_j} + \frac{\partial v_j}{\partial^{t+\Delta t}x_i}\right) \tag{30}$$

Using Eq. (29), the variation of the strain tensor can be obtained as (Gadala et al., 1998):

$$\begin{aligned}\delta\, {}_{t+\Delta t}l_{ij} &= \frac{1}{2}\left(\frac{\partial \delta v_i}{\partial^{t}x_k}\frac{\partial^{t}x_k}{\partial^{t+\Delta t}x_j} + \frac{\partial \delta v_j}{\partial^{t}x_k}\frac{\partial^{t}x_k}{\partial^{t+\Delta t}x_i}\right) \\ &\approx \delta\, {}_{t}l_{ij} - \frac{1}{2}\left(\frac{\partial \hat{u}_k}{\partial^{t}x_j}\frac{\partial \delta v_i}{\partial^{t}x_k} + \frac{\partial \hat{u}_k}{\partial^{t}x_i}\frac{\partial \delta v_j}{\partial^{t}x_k}\right)\end{aligned} \tag{31}$$

The volume element $d\,{}^{t+\Delta t}V$ can be related to $d\,{}^{t}V$ through:

$$d\,{}^{t+\Delta t}V \approx d\,{}^{t}V + \left(d\,{}^{t}V\right)'\Delta t = \left(1+\frac{\partial \hat{u}_k}{\partial^{t}x_k}\right)d\,{}^{t}V \tag{32}$$

Similarly, the boundary surface element $d\,{}^{t+\Delta t}A$ can be related to $d\,{}^{t}A$ through:

$$d\,{}^{t+\Delta t}A \approx d\,{}^{t}A + \left(d\,{}^{t}A\right)'\Delta t = (1 + \frac{\partial \hat{u}_k}{\partial^t x_k} - {}^t n_i l_{ij}\,{}^t n_j) d\,{}^t V \tag{33}$$

where ${}^t n_i$ is the normal vector of $d\,{}^t A$.

Using Eq. (16), the Cauchy stress ${}^{t+\Delta t}\sigma_{ij}$ can be expressed as:

$${}^{t+\Delta t}\sigma_{ij} \approx {}^t\sigma_{ij} + {}^t\sigma'_{ij}\Delta t = {}^t\sigma_{ij} + {}^t\dot{\sigma}_{ij}\Delta t + (\hat{u}_k - u_k)\frac{\partial^t \sigma_{ij}}{\partial^t x_k} \tag{34}$$

Similarly, the water pore pressure ${}^{t+\Delta t}p$ may be expressed in terms of the material time derivative ${}^t\dot{p}$ of pore pressure as follows:

$${}^{t+\Delta t}p \approx {}^t p + {}^t p'\Delta t = {}^t p + {}^t\dot{p}\Delta t + (\hat{u}_k - u_k)\frac{\partial^t p}{\partial^t x_k} \tag{35}$$

The material derivative ${}^{t+\Delta t}\dot{p}$ of pore water pressure may be written as:

$${}^{t+\Delta t}\dot{p} = {}^{t+\Delta t}p' + (v_k - \hat{v}_k)\frac{\partial^{t+\Delta t}p}{\partial^{t+\Delta t}x_k} \approx \frac{{}^{t+\Delta t}p - {}^t p}{\Delta t} + (v_k - \hat{v}_k)\frac{\partial^{t+\Delta t}p}{\partial^{t+\Delta t}x_k} \tag{36}$$

Using Eq. (29), we can get:

$$\frac{\partial^{t+\Delta t}p}{\partial^{t+\Delta t}x_i} = \frac{\partial^{t+\Delta t}p}{\partial^t x_j}\frac{\partial^t x_j}{\partial^{t+\Delta t}x_i} = \frac{\partial^{t+\Delta t}p}{\partial^t x_i} - \frac{\partial \hat{u}_j}{\partial^t x_i}\frac{\partial^{t+\Delta t}p}{\partial^t x_j} \tag{37}$$

From Eq. (20), the mass densities between the configurations at time $t+\Delta t$ and time t may be expressed as:

$${}^{t+\Delta t}\rho \approx {}^t\rho + {}^{t+\Delta t}\rho'\Delta t = {}^t\rho - {}^t\rho\frac{\partial v_i}{\partial^t x_i}\Delta t - \left(u_i - \hat{u}_i\right)\frac{\partial^t \rho}{\partial^t x_j} \tag{38}$$

The equilibrium of body should be satisfied at time $t+\Delta t$, and the weak formulation of equation (4) obtained is:

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\rho a_i \delta v_i d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\delta_{\,t+\Delta t} l_{ij} d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R \tag{39}$$

where ${}^{t+\Delta t}R$ is the external work done by the applied body forces and tractions. Let ${}^{t+\Delta t}B_i$ be the force acceleration per unit volume and ${}^{t+\Delta t}T_i$ the traction, then ${}^{t+\Delta t}R$ is:

$${}^{t+\Delta t}R = \int_{{}^{t+\Delta t}A} {}^{t+\Delta t}T_i \delta v_i d\,{}^{t+\Delta t}A + \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\rho\,{}^{t+\Delta t}B_i \delta v_i d\,{}^{t+\Delta t}V \tag{40}$$

Using the above equations and applying Gauss theorem, the linearized form of the equilibrium equation takes the form:

$$\begin{aligned}
&\int_{{}^tV} {}^t\rho \hat{a}_i \delta v_i d\,{}^tV - \int_{{}^tV} \Delta t\,{}^t\rho \hat{a}_i \frac{\partial v_k}{\partial^t x_k}\delta v_i d\,{}^tV \\
&- \int_{{}^tV} {}^t\rho(\hat{v}_k - v_k)\frac{\partial v_i}{\partial^t x_k}\delta v_i d\,{}^tV \\
&+ \int_{{}^tV} {}^t\sigma_{ij}\delta_t l_{ij} d\,{}^tV + \int_{{}^tV} {}^t\dot{\sigma}_{ij}\Delta t \delta_t l_{ij} d\,{}^tV \\
&+ \int_{{}^tV} {}^t\sigma_{ij}\frac{\partial u_k}{\partial^t x_k}\delta_t l_{ij} d\,{}^tV \\
&- \int_{{}^tV} {}^t\sigma_{ij}\frac{\partial \hat{u}_k}{\partial^t x_j}\delta\left(\frac{\partial u_i}{\partial^t x_k}\right) d\,{}^tV \\
&- \int_{{}^tV} (\hat{u}_k - u_k)\,{}^t\sigma_{ij}\delta\left(\frac{\partial_t l_{ij}}{\partial^t x_k}\right) d\,{}^tV = {}^{t+\Delta t}R
\end{aligned} \tag{41}$$

$$\begin{aligned}
{}^{t+\Delta t}R &= \int_{{}^tV} {}^t\rho({}^tB_i + \Delta B_i)\delta v_i d\,{}^tV \\
&- \int_{{}^tV} {}^t\rho(\hat{u}_k - u_k)\frac{\partial^t B_i}{\partial^t x_k}\delta v_i d\,{}^tV \\
&- \int_{{}^tV} {}^t\rho(\hat{u}_k - u_k)\,{}^tB_i\delta\left(\frac{\partial v_i}{\partial^t x_k}\right) d\,{}^tV \\
&+ \int_{{}^tA} ({}^tT_i + \Delta T_i)\delta v_i d\,{}^tA \\
&+ \int_{{}^tA} {}^tT_i\left(\frac{\partial \hat{u}_k}{\partial^t x_k} - {}^t n_m\,{}_t l_{mn}\,{}^t n_n\right)\delta v_i d\,{}^tA
\end{aligned} \tag{42}$$

Using a kind of finite volume method proposed by Oka et al. (1994) and Akai et al. (1978), and integrating the continuity Eq. (5) for an element volume ${}^{t+\Delta t}V_e$, we obtain:

$$\int_{{}^{t+\Delta t}V_e} \frac{\partial \dot{w}_i}{\partial x_i} d\,{}^{t+\Delta t}V_e + \int_{{}^{t+\Delta t}V_e} {}^{t+\Delta t}l_{ii}\, d\,{}^{t+\Delta t}V_e + \int_{{}^{t+\Delta t}V_e} \frac{n}{K^f}\, {}^{t+\Delta t}\dot{p}\, d\,{}^{t+\Delta t}V_e = 0 \tag{43}$$

where

$$\dot{w}_i = \frac{k}{\gamma_w}\left(-\frac{\partial^{t+\Delta t}p}{\partial^{t+\Delta t}x_i} + \rho^f\, {}^{t+\Delta t}B_i - \rho^f \dot{v}_i\right) \tag{44}$$

Using Gauss theorem, we obtain the linearized form of the continuity equation:

$$\begin{aligned}
&\frac{k}{\gamma_w}\Bigg(\int_{{}^tA_e} \rho^f\, {}^{t+\Delta t}B_i n_i\, d\,{}^tA_e - \int_{{}^tA_e} \frac{\partial^{t+\Delta t}p}{\partial^t x_i} n_i\, d\,{}^tA_e \\
&- \int_{{}^tA_e} \frac{\partial^{t+\Delta t}p}{\partial^t x_k}\frac{\partial \hat{u}_k}{\partial^t x_i} n_i\, d\,{}^tA_e - \int_{{}^tA_e} \rho^f \hat{a}_i n_i\, d\,{}^tA_e \\
&- \int_{{}^tA_e} \rho^f (v_k - \hat{v}_k)\frac{\partial v_i}{\partial^t x_k} n_i\, d\,{}^tA_e\Bigg) \\
&+ \int_{{}^tV_e} \frac{\partial v_i}{\partial^t x_i} d\,{}^tV_e \\
&+ \frac{n}{\Delta t K^f}\int_{{}^tV_e} ({}^{t+\Delta t}p - {}^tp)\left(1 + \frac{\partial \hat{u}_k}{\partial^t x_k}\right) d\,{}^tV_e \\
&+ \frac{n}{K^f}\int_{{}^tV_e} (v_k - \hat{v}_k)\frac{\partial^{t+\Delta t}p}{\partial^t x_k} d\,{}^tV_e = 0
\end{aligned} \tag{45}$$

Equations (41), (42) and (45), together with the constitutive equation Eq. (25), will define a coupled set of ALE formulations for saturated soils. Pore water pressure and two sets of displacements, those of the material points and the mesh grid, are the three basic unknown variables. If the classical isoparametric is considered and the three unknown variables are expressed in their grid point values, the coupled ALE formulations proposed can be discretized by the standard finite element procedure.

6. The Operator-split ALE Method

The coupled ALE equations for saturated soils are complicated and no easily solved. An alternative method is referred to as an operator-split technique (Benson, 1989; Aymone et al., 2001), which is adopted in this section. The operator-split method consists of two steps, a Lagrangian step and an Eulerian step. First, the reference system (finite element mesh) follows the material deformation in the Lagrangian step, and a pure updated Lagrangian procedure step is done. Secondly, mesh smoothing is performed and the reference system is changed as desired. The solution then is remapped from the Lagrangian mesh to the new reference mesh to complete the Eulerian step.

6.1 The Lagrangian step

The Lagrangian step is a classical Lagrangian formulation calculation. On the base of the FE-FD hybrid method (Oka, 1994; Shibata et al. 1991; Akai et al., 1978), a large strain analysis program for saturated soils has been developed using the updated Lagrangian method (Di and Sato, 2004). The Lagrangian step of the operator-split ALE method can be easily implemented in the existing Lagrangian program source codes for large deformation problem.

Using the finite element method, standard finite element approach procedure is used for the equilibrium equation Eq. (39). Finally render the first coupled equation in the matrix form:

$$\mathbf{M}\ddot{\mathbf{u}}_\mathbf{N} + \mathbf{K}\Delta\mathbf{u}_\mathbf{N} + \mathbf{G}\mathbf{p}_\mathbf{EN} = \mathbf{T} \tag{46}$$

where $\mathbf{p}_\mathbf{EN}$ is the excess pore pressure values of elements, $\mathbf{u}_\mathbf{N}$ the displacement vector at the nodes, $\mathbf{M}$ the mass matrix, $\mathbf{K}$ the total stiffness matrix including a material stiffness part and a geometrical one, $\mathbf{G}$ makes up the coupling matrix, and $\mathbf{T}$ is the total load vector.

Employing the FE-FD hybrid method, the second coupled equation in the matrix form is obtained from the continuity equation Eq. (43).

$$\begin{aligned}
&\rho^f \mathbf{G}^\mathbf{T}\ddot{\mathbf{u}}_\mathbf{N} - \frac{\gamma^f}{k}\mathbf{G}^\mathbf{T}\dot{\mathbf{u}}_\mathbf{N} \\
&- \mathbf{H}\mathbf{p}_\mathbf{EN} + \mathbf{A}\dot{\mathbf{p}}_\mathbf{EN} = \mathbf{0}
\end{aligned} \tag{47}$$

6.2 The Eulerian step

(1) Mesh smoothing

Various methods such as h-adaptivity, p-adaptivity and r-adaptivity techniques have been proposed for remeshing structure. The h-adaptivity method changes the mesh connectivity through addition of elements. The p-adaptivity method enhances the polynomial interpolation space in high strain location regions. The r-adaptivity method refines the mesh by relocation of nodes.

Because of large deformation, mesh refinement should be performed at almost every time step. In order to avoid complicated computation, the mesh smoothing scheme in this paper moves nodes as in the r-adaptivity method. Unlike the classical remeshing techniques, the mesh smoothing procedure is performed while the topology of the initial mesh can be preserved. It is explicit, cheap and needs less computational efforts.

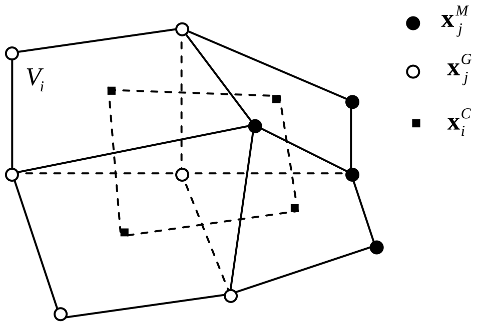

Fig. 2 Relocation of the node

We here decide how to move the reference system. The element nodal pattern is defined by creating a new mesh of the deformed body according to the element shape and the mesh smoothing scheme.

The total volume of elements surrounding a node, V_T, is:

$$V_T = \sum_{i=1}^{n_e} V_i \tag{48}$$

where V_i is the volume of element i, n_e the number of elements surrounding the node. For each element i, the coordinates of its gravity center are:

$$\mathbf{x}_i^C = \frac{1}{n_n}\left(\sum_{j=1}^{n_n^*}\mathbf{x}_j^M + \sum_{j=n_n^*}^{n_n}\mathbf{x}_j^G\right) \tag{49}$$

where n_n is the number of nodes in the element i, $n^*{}_n$ the number of unrelocated nodes in the element i, $\mathbf{x}_j^M$ is the nodal coordinates.

The relocated coordinates of the node then is calculated as (See Fig. 2):

$$\mathbf{x}_i^G = \frac{1}{V_T}\left(\sum_{j=1}^{n_e}\mathbf{x}_j^C V_j\right) \tag{50}$$

The boundaries of the new mesh and those of the old mesh obtained at the end of the updated Lagrangian step must coincide at each step. The boundary points may only move in the tangential direction with a displacement different from the material displacement.

(2) Transferring state variables

While the reference configuration changes to the new mesh pattern, the state variables (stress, strain etc.) obtained in the updated Lagrangian step are frozen. The state variable fields are then remapped onto the new reference configuration (new mesh) from the material configuration (the old mesh). The aim of this step is to solve Eq. (16) by a local least-squares smoothing method in Hinton et al. (1974) and Aymone et al. (2001).

In the updated Lagrangian step, state variable fields of elements are usually calculated at Gauss points. The values should be transferred to the Gauss points of the new finite element mesh (see Fig. 3). Because mesh smoothing is carried out at each time step, the distance between the non-remeshed and remeshed Gauss points is small and any lack of equilibrium on the new mesh can be overcome in the next step. The known values of state variable at Gauss point, $\mathbf{f}_{GP}^M$, can be related to the values at nodes, $\mathbf{f}_N^M$, as:

$$\mathbf{f}_{GP}^M = \sum_{\alpha=1}^{n_n}\left(\phi_\alpha f_{N,\alpha}^M\right) = \boldsymbol{\phi}\mathbf{f}_N^M \tag{51}$$

where $\boldsymbol{\phi}$ is the interpolation function matrix evaluated at Gauss points and $f_{N,\alpha}^M$ the value of a state variable f at the node α of the material configuration.

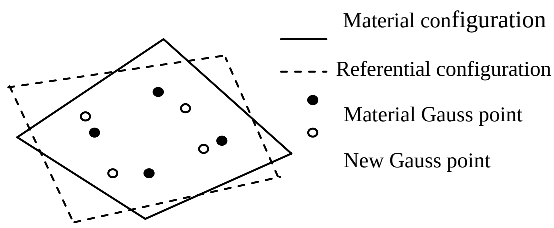

Fig. 3 Gauss points on the material and the new referential configurations

Inverting Eq. (51), the values at nodes are obtained.

$$\mathbf{f}_N^M = \boldsymbol{\phi}^{-1}\mathbf{f}_{GP}^M \tag{52}$$

The gradient of the state variable near the material Gauss point is calculated as:

$$\frac{\partial f_{GP}^{M}}{\partial X_i} = \sum_j \left(\frac{\partial f_{GP}^{M}}{\partial \xi_j} \frac{\partial \xi_j}{\partial X_i} \right) \tag{53}$$

where $\frac{\partial f_{GP}^{M}}{\partial \xi_j} = \sum_{\alpha=1}^{n_n} \left(\frac{\partial \phi_\alpha}{\partial \xi_j} f_{N,\alpha}^{M} \right)$, ϕ_α is the interpolation function, ξ_j the natural coordinates of element.

The term, $\frac{\partial \xi_j}{\partial X_i}$, in Eq. (53), can be calculated by inverting $\frac{\partial X_i}{\partial \xi_j} = \sum_{\alpha=1}^{n_n} \left(\frac{\partial \phi_\alpha}{\partial \xi_j} X_{\alpha,i} \right)$.

Because the distance between the old and new meshes is small, the state variable at Gauss point of the referential configuration is calculated by:

$$f_{GP}^{G} = f_{GP}^{M} + \left(\mathbf{x}_{GP}^{G} - \mathbf{x}_{GP}^{M} \right) \cdot \frac{\partial f_{GP}^{M}}{\partial \mathbf{X}} \tag{54}$$

where $\mathbf{x}_{GP}^{G}$ is the coordinates of Gauss point on new mesh, $\mathbf{x}_{GP}^{M}$ the coordinates of the non-remeshed Gauss point.

All state variables, including the nonlinear path dependent material variables, should be transferred from the old mesh to the new one.

7. Numerical Example

Using updated Lagrangian scheme, a large deformation analysis program for saturated soils has been developed (Di and Sato, 2004). Based on this existing program source code, the operator-split ALE method described in the preceding sections was implemented. With this program, numerical simulation of an embankment subjected to earthquake motion was carried out. The embankment was considered to be a plane strain problem. Infinite elements were used for the lateral boundary sides. The bottom was impermeable while the ground surface a drainage boundary. The fluidal-elasto-plastic constitutive model presented in Section 4 was used for the saturated soils. The initial finite element mesh of the slope is shown in Fig. 4. The input motion is the strong motion record observed at Port Island, Kobe during the Hyogoken-Nanbu earthquake in 1995, as shown in Fig. 5.

When the updated Lagrangian formulation was used to compute the seismic response of the embankment, some element volumes were turned to negative values due to severe distortion at time 5.1s (See Fig. 6). So stability of computation was destroyed and the program has to be stopped.

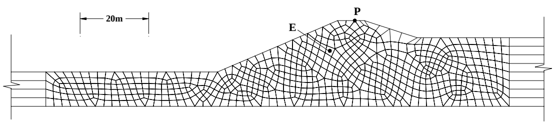

Fig. 4 Initial finite element mesh of an embankment

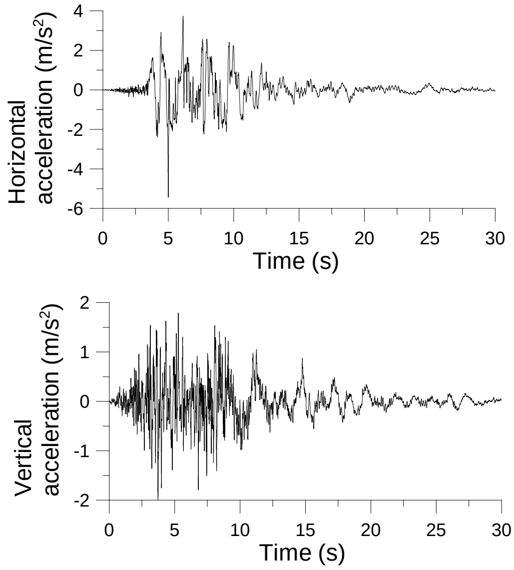

Fig. 5 Input motion

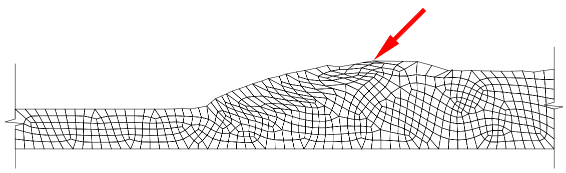

Fig. 6 The deformed mesh at time 5.1s (The updated Lagrangian method)

In the simulation with the arbitrary Lagrangian Eulerian formulation, the finite element mesh was kept smooth at time 5.1s (See Fig. 7) and the calculation could easily be continued. The deformed configurations at time 10.0s and after the earthquake are shown in Figs. 8 and 9, in which the scale of deformation is the same as that of the finite element mesh. The time histories of horizontal and vertical displacement on point P are shown in Fig. 10. The shear stress-strain relationship and stress paths in element E are shown in Fig. 11.

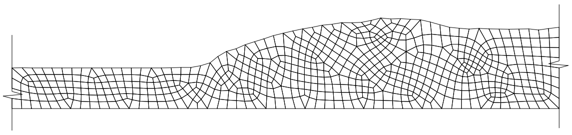

Fig. 7 The deformed mesh at time 5.1s (The ALE method)

8. Conclusions

Due to extensive mesh distortion and elements entanglement in simulating large deformation problem, numerical difficulties and loss in accuracy caused by conventional Lagrangian finite element

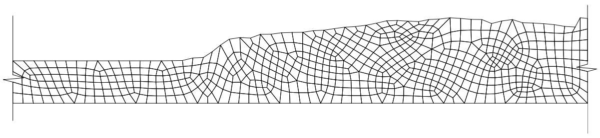

Fig. 9 The deformed mesh after the earthquake (The ALE method)

formulations arise. The ALE finite element method can be utilized to overcome these difficulties.

In this paper, a coupled ALE formulation for saturated soils is derived on the basis of Biot's two-phase theory and an incremental approach. Jaumann stress rate is employed as an objective measure of stress rate to consider large deformation. The grid time derivative of stress is found using the Jaumann stress rate, and a fluidal-elasto-plastic constitutive model of saturated soils can be adopted without further modification in the ALE finite element analysis. The proposed ALE formulations consist of an equilibrium equation and a continuity equation. Pore water pressure and two sets of displacements, those of the material points and the mesh grid, are the three basic unknown variables. The coupled ALE formulation can be discretized by standard finite element procedure.

To an existing program source code written by the updated Lagrangian scheme, the operator-split ALE method is implemented. This algorithm is composed of two steps, a Lagrangian step and an Eulerian step. In the Lagrangian step a classical updated Lagrangian formulation calculation is done, whereas the finite element mesh moves with the material. In the Eulerian step, remeshing of the deformed body is conducted as desired and the solution variables are transferred from the Lagrangian mesh to the new mesh. The Eulerian step is performed using mesh smoothing and data transferring schemes in this paper. The proposed method for large deformation analysis is illustrated by a numerical example of embankment subjected earthquake motion. The example shows that the proposed scheme can overcome numerical difficulties caused by severely distortion and entanglement of elements, which often occurs in

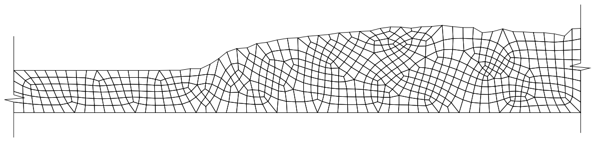

Fig. 8 The deformed mesh at time 10.0s (The ALE method)

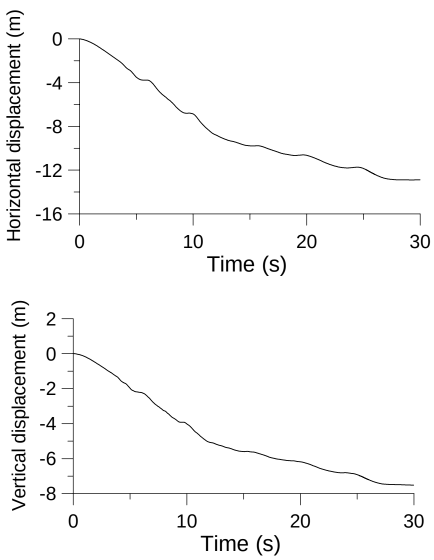

Fig. 10 Time history of displacement on point P

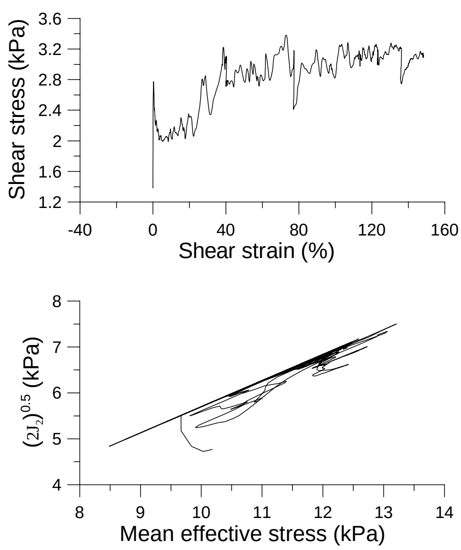

Fig. 11 The shear stress-strain relationship and stress paths in element E

large deformation analyses by the classical updated Lagrangian formulation.

Acknowledgments

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要　旨

水で飽和された土の動的大変形問題を解析するために、不定形ラグランジェ・オイラー（ALE）有限要素法を展開した。流弾塑性構成式を用い、増分形式で支配方程式を表現した上で、連成形式のALE法の定式化を行なった。updated Lagrangian 法で展開されている既存の解析プログラムにALE法をoperator-split形式で導入した。この方法はラグランジェとオイラーステップの２段階からなっている。前者は普通のupdated Lagrangian 法により解析を行い、後者でメッシュの切り替えとデータの変換を行なって、大変形形解析を行うものである。提案する手法の有効性を提体の動的大変形シミュレーションにより検証した。

キーワード: 飽和土、大変形、不定形ラグランジェ・オイラー（ALE）法、有限要素法、液状化、流動化