C09

A Three-Dimensional Model of Subsurface Flow in an Unconfined Surface Soil Layer on an Irregular Hillslope

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To better assess hillslope stability for landslide prediction, we would like to develop a three-dimensional model for shallow groundwater flow in a surface soil layer on an irregular hillslope. In terms of the assumption of shallow groundwater flow, we derived a new and Boussinesq-type perturbation solution of hydraulic head as well as a depth-averaged equation of groundwater table evolution. For numerical solutions, we used the leading-order evolution equation having a strong advection term, a nonlinear diffusion term and a source term. To tackle efficient and accurate calculation efficiency, we proposed a new and high resolution Godunov-type finite volume scheme with specific treatments to the nonlinear diffusion term for assuring the property of numerically well-balancing. Some cases are conducted for verification of the new model we proposed. This work is supposed to provide a new three-dimensional theory of groundwater motion and a corresponding numerical model.

Fundamental theory

Governing equations

We consider a thin and sloping aquifer consisting of isotropic and homogeneous porous medium, and the fluid in the pores is homogeneous and incompressible. The Cartesian coordinates of (x',y',z') is used. The seepage velocity $\boldsymbol{u}' = (u',v',w')$ [m s^{-1}] is expressed by the Darcy's law

$$\boldsymbol{u}' = -k_0\nabla' h', \tag{1}$$

where k_0 is the hydraulic conductivity [m s^{-1}], ∇' is the Laplacian operator, h' is the hydraulic head [m],

$$h' = p'/\gamma_w + z', \tag{2}$$

where p' is the pore water pressure [Pa], γ_w is the water specific weight [N m^{-3}].

For groundwater, the continuity equation reads

$$\nabla'\cdot\boldsymbol{u}' = 0, \quad b' < z < \eta' \text{ and } (x',y') \in \mathcal{D}', \tag{3}$$

where $b'(x',y')$ and $\eta'(x',y',t')$ denote the invariant bottom and phreatic surface, respectively, and $\mathcal{D}'$ is the horizontal boundary. At the bottom, the no-slip condition reads

$$w' = u'\frac{\partial b'}{\partial x'} + v'\frac{\partial b'}{\partial y'}, \quad \text{on } z' = b'. \tag{4}$$

At the phreatic surface, the kinematic boundary condition with a spatially-varying rainfall recharge $I'(x',y',t') > 0$ [m s^{-1}] is imposed as

$$w' = S_s\frac{\partial \eta'}{\partial t'} + u'\frac{\partial \eta'}{\partial x'} + v'\frac{\partial \eta'}{\partial y'} - I', \quad \text{on } z' = \eta', \tag{5}$$

where S_s is the effective porosity [-]; the free-surface dynamic boundary condition with zero-pressure is

$$p' = 0 \text{ and } h' = \eta', \quad \text{on } z' = \eta'. \tag{6}$$

Normalized governing equations

All normalized variables are defined as

$$(x,y) = \frac{1}{L}(x',y'),\ (\eta,b) = \frac{1}{D}(\eta',b'),$$
$$p = \frac{p'}{\gamma_w H},\ t = \frac{t'}{S_s L^2/k_0\Phi},\ \boldsymbol{u} = \frac{\boldsymbol{u}'}{k_0\Phi/L},\ h = \frac{h'}{\Phi}. \tag{7}$$

With (7), the normalized hydraulic head reads

$$h = p + z. \tag{8}$$

Then, normalized governing equation and boundary conditions become

$$\epsilon^2\nabla^2 h + \frac{\partial^2 h}{\partial z^2} = 0, \quad b < z < \eta \text{ and } (x,y) \in \mathcal{D}, \tag{9}$$

$$p = 0 \text{ and } h = \eta, \quad \text{on } z = \eta, \tag{10}$$

$$\frac{\partial h}{\partial z} = -\epsilon^2 \left(\frac{\partial \eta}{\partial t} - \nabla h \cdot \nabla \eta - \gamma\right), \quad \text{on } z = \eta, \tag{11}$$

$$\frac{\partial h}{\partial z} = \epsilon^2 \nabla h \cdot \nabla b, \quad \text{on } z = b, \tag{12}$$

where the normalized horizontal Laplacian operator is

$$\nabla(\cdot) = \left(\frac{\partial \cdot}{\partial x}, \frac{\partial \cdot}{\partial y}\right), \tag{13}$$

and the normalized rainfall recharge and small shallowness parameter for a thin soil layer are

$$\gamma = \frac{I' L^2}{k_0 \Phi H} \text{ and } \epsilon^2 = \left(\frac{D}{L}\right)^2 \ll 1. \tag{14}$$

Perturbation solution

The normalized governing equations, (9) to (12), form a perturbation problem. Using $\epsilon^2 \ll 1$ an infinite series of the hydraulic head is expanded as

$$h = h_0 + \epsilon^2 h_1 + \epsilon^4 h_2 + \cdots, \tag{15}$$

Applying the perturbation method to the governing equations, we obtained a new perturbation solution

$$\begin{aligned} h = \eta + \epsilon^2 [(\eta - z)(\nabla\eta \cdot \nabla b - b\nabla^2 \eta) \\ + \tfrac{1}{2}(\eta^2 - z^2)\nabla^2 \eta] + \mathcal{O}(\epsilon^4), \end{aligned} \tag{16}$$

regarding irregular bottom and rainfall recharge. Equation (16) shows that the leading-order solution η is independent of z. With (16), the velocities are

$$\begin{aligned} (u, v) = -\nabla h = -\nabla\eta - \epsilon^2 \{\nabla\eta(\nabla\eta \cdot \nabla b) \\ + \nabla\eta(\eta - b)\nabla^2\eta + \nabla(\nabla\eta \cdot \nabla b - b\nabla^2 h) \\ + \tfrac{1}{2}(\eta^2 - z^2)\nabla^2 \nabla\eta\} + \mathcal{O}(\epsilon^4), \end{aligned} \tag{17}$$

$$\begin{aligned} w &= -\frac{\partial h}{\partial z} \\ &= -\epsilon^2 [(b - z)\nabla^2 \eta - \nabla\eta \cdot \nabla b] + \mathcal{O}(\epsilon^4). \end{aligned} \tag{18}$$

The leading-order vertical velocity is zero, and this verifies the shallow flow assumption.

Equation of phreatic surface evolution

Applying the depth-averaging method to continuity (9) with other boundary conditions yields the equation of phreatic surface evolution as

$$\frac{\partial \eta}{\partial t} = [\nabla \cdot (\eta - b)\nabla\eta + \gamma] + + \mathcal{O}(\epsilon^2). \tag{19}$$

Equation (19) is verified to be equal to the classical solutions (Parlange et al., 1984; Chen and Liu, 1995).

Numerical scheme

To find numerical solutions, we used the leading-order equation of groundwater depth evolution, as below

$$\frac{\partial H}{\partial t} = \nabla \cdot H\nabla(H + b) + \gamma, \tag{20}$$

where $H = \eta - b$ is the total groundwater depth [m]. Equation (20) is a nonlinear advection-diffusion equation with a source term. To achieve efficient computation, an explicit scheme is required. With specific treatments for the nonlinear diffusion term and for assuring well-balancing property, a new and high resolution Godunov scheme (LeVeque, 2002; Busto et al., 2016) is proposed to numerically solve (20).

Expected Results

A new complete solution of groundwater motion in a sloping unconfined aquifer with a spatial-varying rainfall recharge has been derived. A new scheme for solving the equation of groundwater depth evolution is also proposed. Some cases with real three-dimensional topography will be conducted. This work can benefit efficient and accurate calculation of three-dimensional groundwater motion in a thin and unconfined sloping aquifer under any given rainfall.

References

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